4.2 Proving and Applying the Sine and Cosine Laws for Obtuse Triangles

YOU WILL NEED
- calculator
- ruler

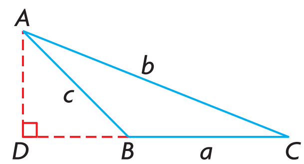
EXPLORE…
- An isosceles obtuse triangle has one angle that measures 120° and one side length that is 5 m. What could the other side lengths be?

GOAL

Explain steps in the proof of the sine and cosine laws for obtuse triangles, and apply these laws to situations that involve obtuse triangles.

INVESTIGATE the Math

In Lesson 3.2, you analyzed Ben's proof of the sine law for acute triangles. Ben wanted to adjust his proof to show that the sine law also applies to obtuse triangles. Consider Ben's new proof:



Step 1 I drew obtuse triangle ABC with height AD.

Step 2 I wrote equations for $\sin(180° - \angle ABC)$ and $\sin C$ using the two right triangles.

In $\triangle ABD$,

$$\sin(180° - \angle ABC) = \frac{\text{opposite}}{\text{hypotenuse}}$$
$$\sin(180° - \angle ABC) = \frac{AD}{c}$$
$$c \sin(180° - \angle ABC) = AD$$
$$c \sin \angle ABC = AD$$

In $\triangle ACD$,

$$\sin C = \frac{\text{opposite}}{\text{hypotenuse}}$$
$$\sin C = \frac{AD}{b}$$
$$b \sin C = AD$$

Step 3 Both expressions for AD equal each other (transitive property), so:

$$c \sin \angle ABC = b \sin C$$
$$\frac{(c \sin \angle ABC)}{\sin C} = b$$
$$\frac{c}{\sin C} = \frac{b}{\sin \angle ABC}$$

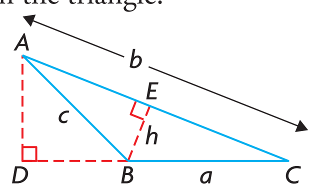
Step 4 I drew a new height, h, from B to base b in the triangle.

In $\triangle ABE$,

$$\sin A = \frac{h}{c}$$
$$c \sin A = h$$

In $\triangle CBE$,

$$\sin C = \frac{h}{a}$$
$$a \sin C = h$$



Step 5 Both expressions for h equal each other, so:

$$c \sin A = a \sin C$$
$$\frac{c}{\sin C} = \frac{a}{\sin A}$$

I have already shown that

$$\frac{c}{\sin C} = \frac{b}{\sin \angle ABC}, \text{ so}$$

$$\frac{c}{\sin C} = \frac{b}{\sin \angle ABC} = \frac{a}{\sin A}$$

? How can you explain what Ben did to prove the sine law for obtuse triangles?

A. Why did Ben choose to write expressions for the sin $(180° - \angle ABC)$ and sin C?

B. In step 3, Ben mentions the transitive property. What is this property, and how did he use it in this step?

C. In step 4, Ben drew a new height in $\triangle ABC$. Why was this necessary?

D. Why was Ben able to equate all three side angle ratios in step 5?

Reflecting

E. Compare the proof above to Ben's original proof in Lesson 3.2, pages 118 to 119. How is the proof of the sine law for obtuse triangles the same as that for acute triangles? How is it different?

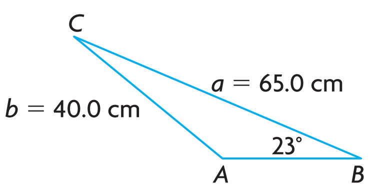
F. If Ben started his proof by writing expressions for sin $(180° - \angle CBA)$ and sin A, where would he have drawn the height in step 1?

APPLY the Math

EXAMPLE 1 Use reasoning and the sine law to determine the measure of an obtuse angle

In an obtuse triangle, $\angle B$ measures 23.0° and its opposite side, b, has a length of 40.0 cm. Side a is the longest side of the triangle, with a length of 65.0 cm. Determine the measure of $\angle A$ to the nearest tenth of a degree.

Bijan's Solution



I drew an obtuse triangle to represent $\triangle ABC$.

I knew that the longest side is always opposite the largest angle, so the 65.0 cm side must be opposite the obtuse angle, $\angle A$.

Since $\triangle ABC$ is not a right triangle, I knew that I could not use the primary trigonometric ratios to determine the measure of $\angle A$.

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

I noticed that the diagram has two side-angle pairs with only one unknown, ∠*A*. I decided to use the sine law.

$$\frac{\sin A}{65.0} = \frac{\sin 23°}{40.0}$$

The measure of an angle is the unknown, so I used the form of the sine law that has the angles in the numerator.

$$65.0\left(\frac{\sin A}{65.0}\right) = \left(\frac{\sin 23°}{40.0}\right)65.0$$
$$\sin A = 0.6349...$$

I isolated sin *A*.

$$\angle A = \sin^{-1}(0.6349...)$$
$$\angle A = 39.4153...°$$

I used the inverse sine to determine the measure of ∠*A*.

$$\angle A = 180° - 39.4153...°$$
$$\angle A = 140.5846...°$$

My reasoning suggests that ∠*A* must be the obtuse angle. I used the relationship $\sin A = \sin(180° - A)$.

∠*A* measures 140.6°.

The measure of the angle seems appropriate, according to my diagram.

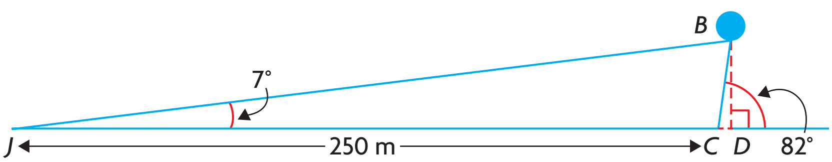
Your Turn

Determine the length of side *AB* in △*ABC* above, to the nearest tenth of a centimetre.

EXAMPLE 2 Solving a problem using the sine law

Colleen and Juan observed a tethered balloon advertising the opening of a new fitness centre. They were 250 m apart, joined by a line that passed directly below the balloon, and were on the same side of the balloon. Juan observed the balloon at an angle of elevation of 7° while Colleen observed the balloon at an angle of elevation of 82°. Determine the height of the balloon to the nearest metre.

Colleen's Solution



I drew a diagram to represent the situation. The height of the balloon is represented by *BD*. I need to determine the length of *BC* in order to determine the length of *BD*. I can use the sine law in Δ*BJC*.

$\angle BCJ = 180° - 82°$
$\angle BCJ = 98°$

I determined the supplement of 82° to determine the measure of a second angle in $\triangle BJC$. This is an obtuse triangle.

$\angle JBC = 180° - 98° - 7°$
$\angle JBC = 75°$

I determined the measure of the third angle in ΔBJC. This gave me a known side, JC, and a known angle opposite this side, $\angle JBC$, in this triangle.

$$\frac{BC}{\sin \angle BJC} = \frac{JC}{\sin \angle JBC}$$

I used the sine law to write an equation that involved BC and the known side-angle pair.

$$\frac{BC}{\sin(7°)} = \frac{250}{\sin(75°)}$$
$$BC = \sin(7°)\left(\frac{250}{\sin(75°)}\right)$$
$$BC = 31.542\ldots$$

I substituted the known information into the equation and solved for BC.

$$\sin(\angle BCD) = \frac{BD}{BC}$$

I wrote an equation that involved BD, BC, and the known angle in ΔBCD.

$$\sin(82°) = \frac{BD}{31.542\ldots}$$
$$(31.542\ldots)(\sin(82°)) = BD$$
$$31.235\ldots \text{ m} = BD$$

I substituted the known information into the equation and solved for BD.

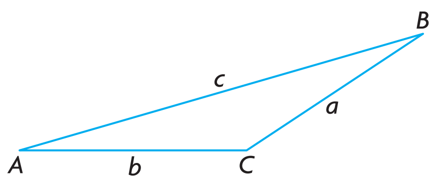
The advertising balloon is 31 m above the ground.

Your Turn

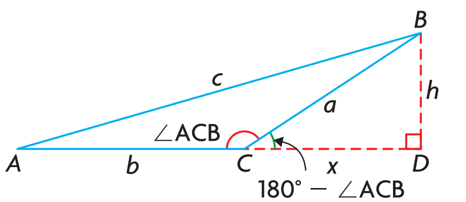
Determine the distance between Juan and the balloon.

EXAMPLE 3 Use reasoning to demonstrate the cosine law for obtuse triangles

Show that the cosine law holds for obtuse triangles, using $\triangle ABC$.



Hyun Yoon's Solution



I extended the base of the triangle to D. This created two overlapping right triangles, ΔCBD and $\triangle ABD$, with height BD. It also created two angles at C, $\angle ACB$ and $\angle DCB$, such that $\angle DCB = 180° - \angle ACB$.

In $\triangle ABD$ In $\triangle CBD$

$h^2 = c^2 - (b + x)^2$ $h^2 = a^2 - x^2$

I used the Pythagorean theorem to write two expressions for h^2, using the two right triangles.

The expressions that equal h^2 equal each other (transitive property).

I solved for c^2.

$$c^2 - (b + x)^2 = a^2 - x^2$$
$$c^2 = (b + x)^2 + a^2 - x^2$$
$$c^2 = b^2 + 2bx + x^2 + a^2 - x^2$$
$$c^2 = a^2 + b^2 + 2bx$$

$$\cos(180° - \angle ACB) = \frac{x}{a}$$
$$a\cos(180° - \angle ACB) = x$$

The acute angle in $\triangle CBD$ has a measure of $180° - \angle ACB$.

I used the cosine ratio to write an expression for x.

$$c^2 = a^2 + b^2 + 2b[a\cos(180° - \angle ACB)]$$

$$c^2 = a^2 + b^2 - 2ab\cos\angle ACB$$

I have demonstrated the cosine law.

I substituted my expression for x into my equation.

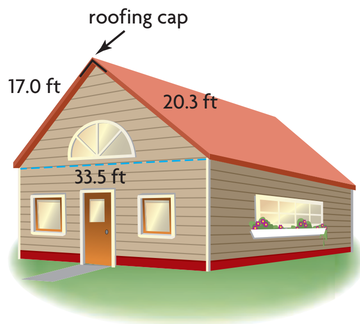
To write an equation that contained only measures found in the original triangle, I used the following fact: $\cos(180° - \angle ACB) = -\cos\angle ACB$

Your Turn

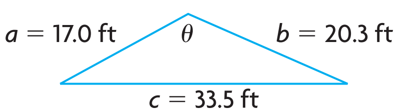
Review the proof of the cosine law for acute triangles in Lesson 3.3, pages 130 to 131. Explain how Hyun Yoon modified this proof to deal with an obtuse triangle.

EXAMPLE 4 Using reasoning and the cosine law to determine the measure of an obtuse angle

The roof of a house consists of two slanted sections, as shown. A roofing cap is being made to fit the crown of the roof, where the two slanted sections meet. Determine the measure of the angle needed for the roofing cap, to the nearest tenth of a degree.



Maddy's Solution: Substituting into the cosine law and then rearranging



I sketched a triangle to represent the problem situation.

The largest angle is θ, because it is opposite the longest side.

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

Three side lengths are given, so I knew that I could use the cosine law.

$$(33.5)^2 = (17.0)^2 + (20.3)^2 - 2(17.0)(20.3)\cos\theta$$

$$(33.5)^2 - (17.0)^2 - (20.3)^2 = -2(17.0)(20.3)\cos\theta$$

$$1122.25 - 289 - 412.09 = -690.2\cos\theta$$

I substituted the known values into the formula for the cosine law and isolated θ.

$$\frac{421.16}{-690.2} = \cos\theta$$

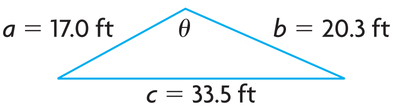
$$\cos^{-1}\left(-\frac{421.16}{690.2}\right) = \theta$$

$$127.6039\ldots^\circ = \theta$$

An angle of 127.6° is needed for the roofing cap.

My answer is reasonable, given the diagram.

Georgia's Solution: Rearranging the cosine law before substituting



I sketched a triangle to represent the problem situation.

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

I knew the lengths of all three sides, so I used the cosine law.

$$c^2 + 2ab\cos\theta = a^2 + b^2 - 2ab\cos\theta + 2ab\cos\theta$$

$$c^2 - c^2 + 2ab\cos\theta = a^2 + b^2 - c^2$$

$$\frac{2ab\cos\theta}{2ab} = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\cos\theta = \frac{a^2 + b^2 - c^2}{2ab}$$

Since I wanted to solve for θ, I rearranged the formula to isolate $\cos\theta$.

$$\cos\theta = \frac{(17.0)^2 + (20.3)^2 - (33.5)^2}{2(17.0)(20.3)}$$

I substituted the values of a, b, and c into the rearranged formula.

$$\cos\theta = -0.6101\ldots$$

$$\theta = \cos^{-1}(-0.6101\ldots)$$

$$\theta = 127.6039\ldots^\circ$$

The angle for the roofing cap should measure 127.6°.

Your Turn

Determine the angle of elevation for each roof section, to the nearest tenth of a degree.

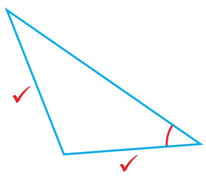
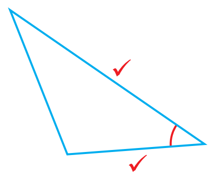
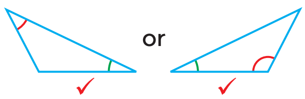
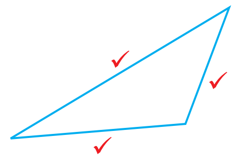
In Summary

Key Idea

- The sine law and cosine law can be used to determine unknown side lengths and angle measures in obtuse triangles.

Need to Know

- The sine law and cosine law are used with obtuse triangles in the same way that they are used with acute triangles.

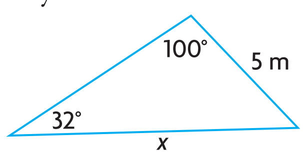
Use the sine law when you know …	Use the cosine law when you know …
- the lengths of two sides and the measure of the angle that is opposite a known side 	- the lengths of two sides and the measure of the contained angle 
- the measures of two angles and the length of any side 	- the lengths of all three sides 

- Be careful when using the sine law to determine the measure of an angle. The inverse sine of a ratio always gives an acute angle, but the supplementary angle has the same ratio. You must decide whether the acute angle, θ, or the obtuse angle, $180° - \theta$, is the correct angle for your triangle.
- Because the cosine ratios for an angle and its supplement are not equal (they are opposites), the measures of the angles determined using the cosine law are always correct.

CHECK Your Understanding

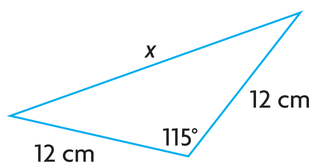
1. There are errors in each application of the sine law or cosine law. Identify the errors.

a)



$$\frac{5}{\sin 100°} = \frac{x}{\sin 32°}$$

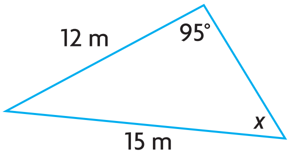
b)



$$12^2 = x^2 + 12^2 - 2(12)(x) \cos 115°$$

2. Which law could be used to determine the unknown angle measure or side length in each triangle? For your answer, choose one of the following: sine law, cosine law, both, neither. Explain your choice.

a)



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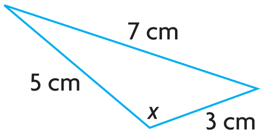

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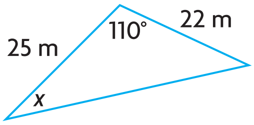

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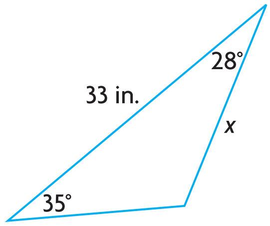

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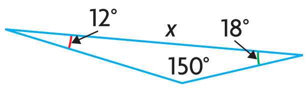
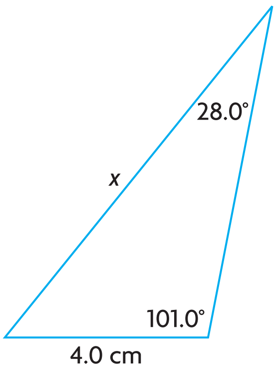


PRACTISING

3. Determine the unknown side length in each triangle, to the nearest tenth of a centimetre.

a)



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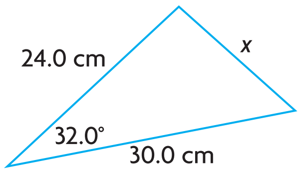

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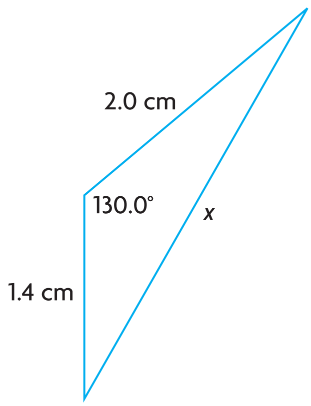
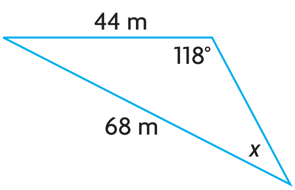


4. Determine the unknown angle measure in each triangle, to the nearest degree.

a)



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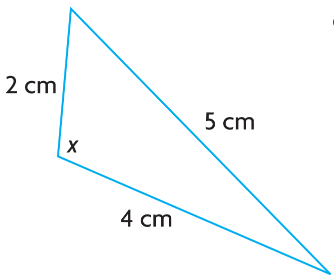

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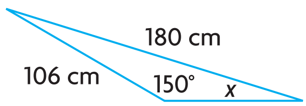
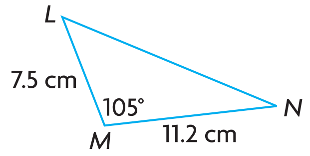


5. Determine each unknown angle measure to the nearest degree and each unknown side length to the nearest tenth of a centimetre.

a)



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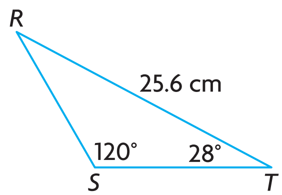

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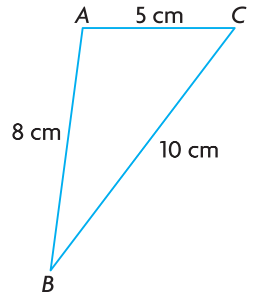

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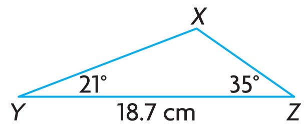


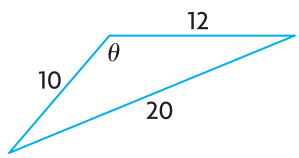
6. A triangle has side lengths of 4.0 cm, 6.4 cm, and 9.8 cm.

a) Sketch the triangle, and estimate the measure of the largest angle.

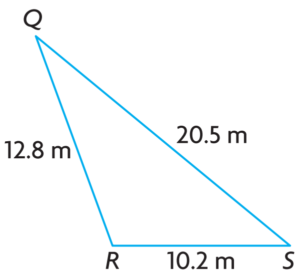
b) Calculate the measure of the largest angle to the nearest tenth of a degree.

c) How close was your estimate to the angle measure you calculated? How could you improve similar estimates in the future?

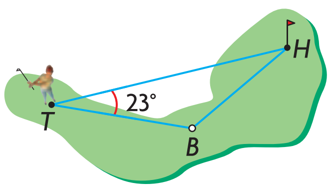
7. Wei-Ting made a mistake when using the cosine law to determine the unknown angle measure below. Identify the cause of the error message on her calculator. Then determine θ to the nearest tenth of a degree.



$$\begin{aligned}
20^2 &= 10^2 + 12^2 - 2(10)(12)\cos\theta \\
400 &= 100 + 144 - 240\cos\theta \\
400 &= 244 - 240\cos\theta \\
400 &= 4\cos\theta \\
100 &= \cos\theta \\
\cos^{-1}(100) &= \theta \\
\text{<error!>} &= \theta
\end{aligned}$$



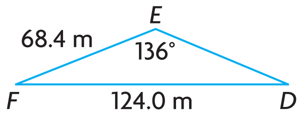
8. In $\triangle QRS$, $q = 10.2$ m, $r = 20.5$ m, and $s = 12.8$ m. Solve $\triangle QRS$ by determining the measure of each angle to the nearest tenth of a degree.

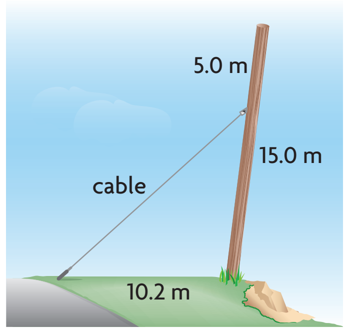


9. While golfing, Sahar hits a tee shot from T toward a hole at H. Sahar hits the ball at an angle of 23° to the hole and it lands at B. The scorecard says that H is 295 yd from T. Sahar walks 175 yd to her ball. How far, to the nearest yard, is her ball from the hole?

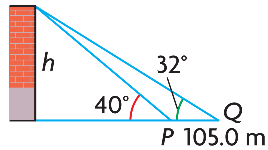
10. The posts of a hockey goal are 6 ft apart. A player attempts to score by shooting the puck along the ice from a point that is 21 ft from one post and 26 ft from the other post. Within what angle, θ, must the shot be made? Express your answer to the nearest tenth of a degree.

11. In $\triangle DEF$, $\angle E = 136°$, $e = 124.0$ m, and $d = 68.4$ m. Solve the triangle. Round each angle measure or side length to the nearest tenth.



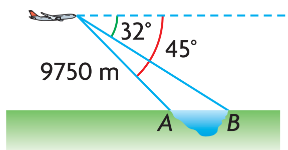


12. A 15.0 m telephone pole is beginning to lean as the soil erodes. A cable is attached 5.0 m from the top of the pole to prevent the pole from leaning any farther. The cable is secured 10.2 m from the base of the pole. Determine the length of the cable that is needed if the pole is already leaning 7° from the vertical.



13. A building is observed from two points, P and Q, that are 105.0 m apart. The angles of elevation at P and Q measure 40° and 32°, as shown. Determine the height, h, of the building to the nearest tenth of a metre.

14. A surveyor in an airplane observes that the angles of depression to points A and B, on opposite shores of a lake, measure 32° and 45°, as shown. Determine the width of the lake, AB, to the nearest metre.



Closing

15. In $\triangle PQR$, $\angle Q$ is obtuse, $\angle R = 12°$, $q = 15.0$ m, and $r = 10.0$ m. Explain to a classmate the steps required to determine the measure of $\angle Q$.

Extending

16. Two roads intersect at an angle of 15°. Darryl is standing on one of the roads, 270 m from the intersection.
    a) Create a problem that must be solved using the sine law. Include a sketch and a solution.
    b) Create a problem that must be solved using the cosine law. Include a sketch and a solution.

17. The interior angles of a triangle measure 120°, 40°, and 20°. The longest side of the triangle is 10 cm longer than the shortest side. Determine the perimeter of the triangle, to the nearest centimetre.