3.2 Proving and Applying the Sine Law

YOU WILL NEED

- ruler
- protractor
- calculator

GOAL

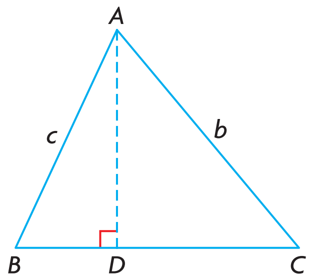
Explain the steps used to prove the sine law. Use the law to solve triangles.

EXPLORE...

- The angles in an acute triangle measure 40°, 55°, and 85°. Could two of the side lengths be 5 cm and 4 cm? Explain.

INVESTIGATE the Math

In Lesson 3.1, you discovered a side–angle relationship in acute triangles. Before this relationship can be used to solve problems, it must be proven to work in all acute triangles. Consider Ben's **proof**:



Step 1
I drew an acute triangle with height *AD*.

In $\triangle ABD$,

$$\sin B = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\sin B = \frac{AD}{c}$$

$$c \sin B = AD$$

In $\triangle ACD$,

$$\sin C = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\sin C = \frac{AD}{b}$$

$$b \sin C = AD$$

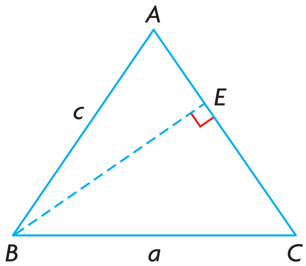
Step 2
I wrote equations for the sine of $\angle B$ and the sine of $\angle C$ in the two right triangles.

$$c \sin B = b \sin C$$

$$\frac{c \sin B}{\sin C} = b$$

$$\frac{c}{\sin C} = \frac{b}{\sin B}$$

Step 3
I set the expressions for *AD* equal to each other.



In $\triangle ABE$,

$$\sin A = \frac{BE}{c}$$

$$c \sin A = BE$$

In $\triangle CBE$,

$$\sin C = \frac{BE}{a}$$

$$a \sin C = BE$$

Step 4
I had expressions that involved sides b and c and $\angle B$ and $\angle C$, but I also needed an expression that involved a and $\angle A$. I drew a height from B to AC and developed two expressions for BE.

$$c \sin A = a \sin C$$

$$c = \frac{a \sin C}{\sin A}$$

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$

Step 5
I set the expressions for BE equal to each other.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Step 6
I set all three ratios equal to each other.

? How can you improve Ben's explanation of his proof?

A. Work with a partner to explain why Ben drew height AD in step 1.

B. In step 2, he created two different expressions that involved AD. Explain why.

C. Explain why he was able to set the expressions for AD equal in step 3.

D. Explain what Ben did to rewrite the equation in step 3.

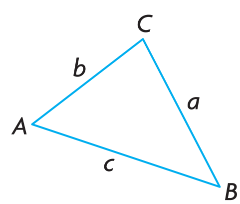
E. In steps 4 and 5, Ben drew a different height BE and repeated steps 2 and 3 for the right triangles this created. Explain why.

F. Explain why he was able to equate all three ratios in step 6 to create the **sine law**.

sine law

In any acute triangle,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



Reflecting

G. Why did Ben not use the cosine ratio or tangent ratio to describe the heights of his acute triangle?

H. If Ben drew a perpendicular line segment from vertex C to side AB, which pair of ratios in the sine law do you think he could show to be equal?

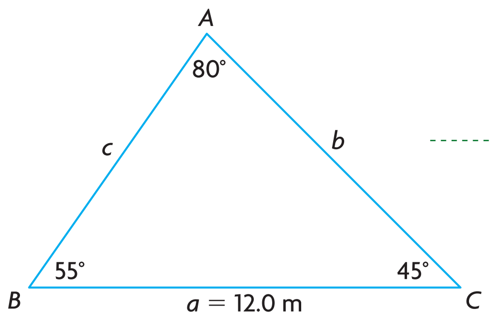
I. Why does it make sense that the sine law can also be written in the form $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$?

APPLY the Math

EXAMPLE 1 Using reasoning to determine the length of a side

A triangle has angles measuring 80° and 55°. The side opposite the 80° angle is 12.0 m in length. Determine the length of the side opposite the 55° angle to the nearest tenth of a metre.

Elizabeth's Solution



I named the triangle *ABC* and decided that the 80° angle was ∠*A*. Then I sketched the triangle, including all of the information available.

I knew that the third angle, ∠*C*, had to measure 45°, because the angles of a triangle add to 180°. I needed to determine *b*.

Since the triangle does not contain a right angle, I couldn't use the primary trigonometric ratios.

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{12.0}{\sin 80°} = \frac{b}{\sin 55°}$$

$$\sin 55°\left(\frac{12.0}{\sin 80°}\right) = \sin 55°\left(\frac{b}{\sin 55°}\right)$$

$$\sin 55°\left(\frac{12.0}{\sin 80°}\right) = b$$

$$9.981... = b$$

I could use the sine law if I knew an opposite side–angle pair, plus one more side or angle in the triangle. I knew *a* and ∠*A* and I wanted to know *b*, so I related *a*, *b*, sin *A*, and sin *B* using $\frac{a}{\sin A} = \frac{b}{\sin B}$. Since *b* was in the numerator, I could multiply both sides by sin 55° to solve for *b*.

The length of *AC* is 10.0 m.

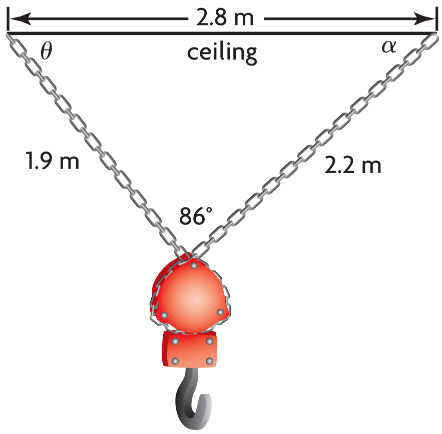
I rounded to the nearest tenth. It made sense that the length of *AC* is shorter than the length of *BC*, since the measure of ∠*B* is less than the measure of ∠*A*.

Your Turn

Using △*ABC* above, determine the length of *AB* to the nearest tenth of a metre.

EXAMPLE 2 Solving a problem using the sine law

Toby uses chains attached to hooks on the ceiling and a winch to lift engines at his father's garage. The chains, the winch, and the ceiling are arranged as shown. Toby solved the triangle using the sine law to determine the angle that each chain makes with the ceiling to the nearest degree. He claims that $\theta = 40°$ and $\alpha = 54°$. Is he correct? Explain, and make any necessary corrections.



Sanjay's Solution

I know Toby's calculations are incorrect, since α must be the smallest angle in the triangle.

In any triangle, the shortest side is across from the smallest angle. Since 1.9 m is the shortest side, $\alpha < \theta$. Toby's values do not meet this condition.

$$\frac{\sin \alpha}{1.9} = \frac{\sin 86°}{2.8}$$

To correct the error, I used the sine law to determine α.

$$1.9\left(\frac{\sin \alpha}{1.9}\right) = 1.9\left(\frac{\sin 86°}{2.8}\right)$$

$$\sin \alpha = 1.9\left(\frac{\sin 86°}{2.8}\right)$$

$$\sin \alpha = 0.6769...$$

$$\alpha = \sin^{-1}(0.6769...)$$

$$\alpha = 42.603...°$$

I multiplied both sides by 1.9 to solve for $\sin \alpha$. Then I evaluated the right side of the equation.

$$\theta = 180° - 86° - 42.603...°$$

$$\theta = 51.396...°$$

I used the fact that angles in a triangle add to 180° to determine θ.

Toby was incorrect. The correct measures of the angles are:

$$\alpha \doteq 43°$$

and

$$\theta \doteq 51°$$

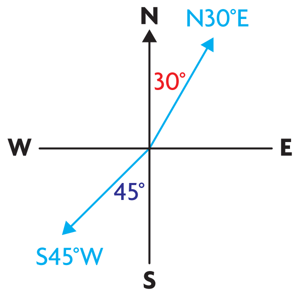
My determinations are reasonable, because the shortest side is opposite the smallest angle.

Communication *Tip*

Greek letters are often used as variables to represent the measures of unknown angles. The most commonly used letters are θ (theta), α (alpha), β (beta), and γ (gamma).

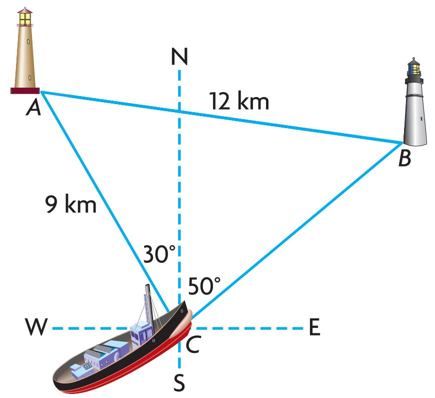
Communication | *Tip*

Directions are often stated in terms of north and south on a compass. For example, N30°E means travelling in a direction 30° east of north. S45°W means travelling in a direction 45° west of south.

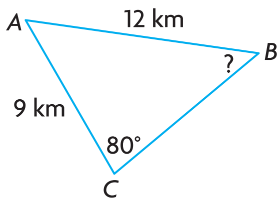


EXAMPLE 3 Using reasoning to determine the measure of an angle

The captain of a small boat is delivering supplies to two lighthouses, as shown. His compass indicates that the lighthouse to his left is located at N30°W and the lighthouse to his right is located at N50°E. Determine the compass direction he must follow when he leaves lighthouse B for lighthouse A.



Anthony's Solution



I drew a diagram. I labelled the sides of the triangle I knew and the angle I wanted to determine.

$$\frac{\sin B}{AC} = \frac{\sin C}{AB}$$

I knew AC, AB, and $\angle C$, and I wanted to determine $\angle B$. So I used the sine law that includes these four quantities.

I used the proportion with sin B and sin C in the numerators so the unknown would be in the numerator.

$$\frac{\sin B}{9} = \frac{\sin 80°}{12}$$

$$9\left(\frac{\sin B}{9}\right) = 9\left(\frac{\sin 80°}{12}\right)$$

$$\sin B = 9\left(\frac{\sin 80°}{12}\right)$$

$$\sin B = 0.7386...$$

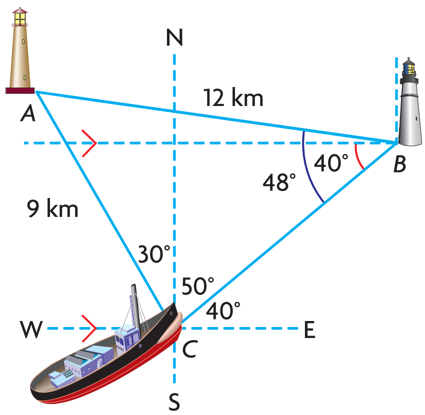
$$\angle B = \sin^{-1}(0.7386...)$$

$$\angle B = 47.612...°$$

I substituted the given information and then solved for sin *B*.

The measure of $\angle B$ is 48°.

The answer seems reasonable. $\angle B$ must be less than 80°, because 9 km is less than 12 km.



I drew a diagram and marked the angles I knew. I knew east-west lines are all parallel, so the alternate interior angle at *B* must be 40°.

The captain must head N82°W from lighthouse *B*.

The line segment from lighthouse *B* to lighthouse *A* makes an 8° angle with west-east. I subtracted this from 90° to determine the direction west of north.

The Fisgard Lighthouse in Victoria, British Columbia, was the first lighthouse built on Canada's west coast and is still in operation today.

Your Turn

In $\triangle ABC$ above, *CB* is about 9.6 km. Use the sine law to determine $\angle A$. Verify your answer by determining the sum of the angles.

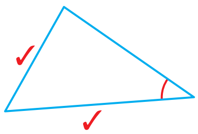
In Summary

Key Idea

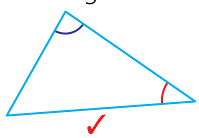
- The sine law can be used to determine unknown side lengths or angle measures in acute triangles.

Need to Know

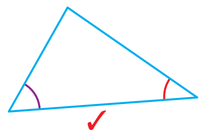
- You can use the sine law to solve a problem modelled by an acute triangle when you know:
  - two sides and the angle opposite a known side.



  - two angles and any side.



or



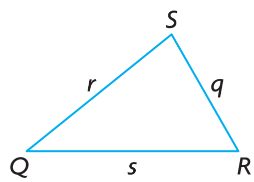
- If you know the measures of two angles in a triangle, you can determine the third angle because the angles must add to 180°.
- When determining side lengths, it is more convenient to use:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

- When determining angles, it is more convenient to use:

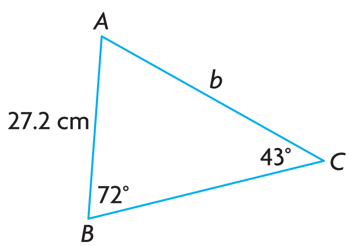
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

CHECK Your Understanding

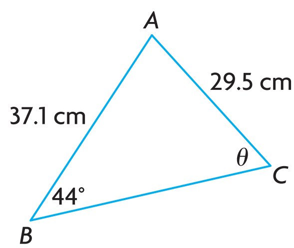


1. Write three equivalent ratios using the sides and angles in the triangle at the right.

2. a) Determine length b to the nearest tenth of a centimetre.



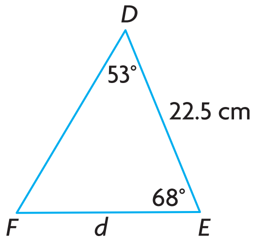
b) Determine the measure of θ to the nearest degree.



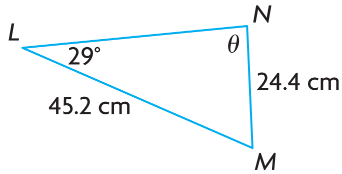
PRACTISING

3. Determine the indicated side lengths to the nearest tenth of a unit and the indicated angle measures to the nearest degree.

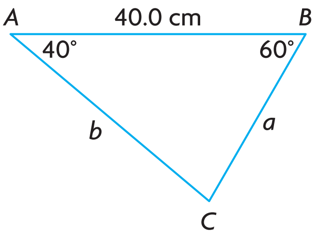
a)



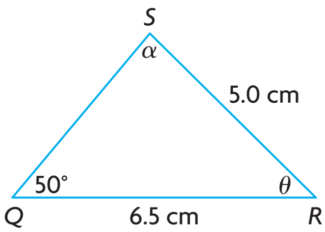
d)



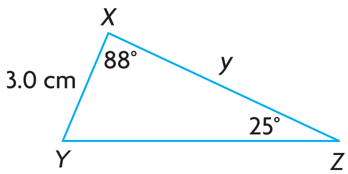
b)



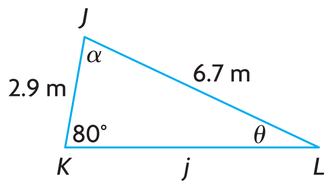
e)



c)



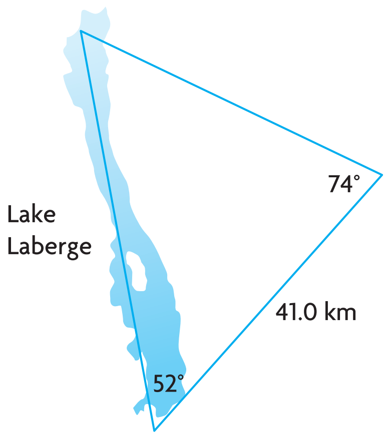
f)



4. Scott is studying the effects of environmental changes on fish populations in his summer job. As part of his research, he needs to know the distance between two points on Lake Laberge, Yukon. Scott makes the measurements shown and uses the sine law to determine the lake's length as 36.0 km.

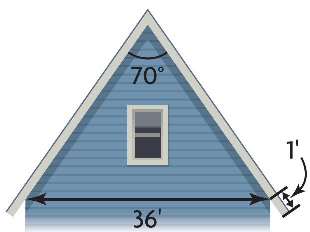
a) Agathe, Scott's research partner, says that his answer is incorrect. Explain how she knows.

b) Determine the distance between the two points to the nearest tenth of a kilometre.

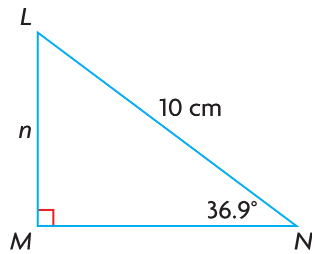


The acidity of northern lakes may be affected by acid rain and snow caused by development.

5. An architect designed a house and must give more instructions to the builders. The rafters that hold up the roof are equal in length. The rafters extend beyond the supporting wall as shown. How long are the rafters? Express your answer to the nearest inch.



6. Draw a labelled diagram for each triangle. Then determine the required side length or angle measure.
   a) In $\triangle SUN$, $n = 58$ cm, $\angle N = 38°$, and $\angle U = 72°$. Determine the length of side u to the nearest centimetre.
   b) In $\triangle PQR$, $\angle R = 73°$, $\angle Q = 32°$, and $r = 23$ cm. Determine the length of side q to the nearest centimetre.
   c) In $\triangle TAM$, $t = 8$ cm, $m = 6$ cm, and $\angle T = 65°$. Determine the measure of $\angle M$ to the nearest degree.
   d) In $\triangle WXY$, $w = 12.0$ cm, $y = 10.5$ cm, and $\angle W = 60°$. Determine the measure of $\angle Y$ to the nearest degree.
7. In $\triangle CAT$, $\angle C = 32°$, $\angle T = 81°$, and $c = 24.1$ m. Solve the triangle. Round sides to the nearest tenth of a metre.
8. a) Determine the value of n to the nearest tenth using
      i) a primary trigonometric ratio.
      ii) the sine law.
   b) Explain why your answers for part a) are the same. Do others in your class agree with your explanation?

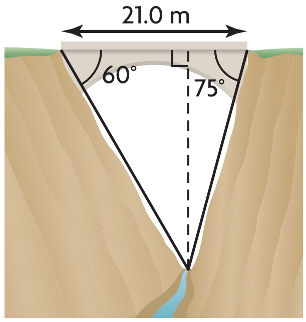


9. Janice is sailing from Gimli on Lake Winnipeg to Grand Beach. She had planned to sail 26.0 km in the direction S71°E; however, the wind and current pushed her off course. After several hours, she discovered that she had actually been sailing S79°E. She checked her map and saw that she must sail S18°W to reach Grand Beach. Determine, to the nearest tenth of a kilometre, the distance remaining to Grand Beach.
   a) Draw a diagram of this situation, then compare it with a classmate's. Make any adjustments needed in your diagrams.
   b) Solve the problem.

Gimli, Manitoba, is a fishing village with a rich Icelandic heritage.

10. A telephone pole is supported by two wires on opposite sides. At the top of the pole, the wires form an angle of 60°. On the ground, the ends of the wires are 15.0 m apart. One wire makes a 45° angle with the ground.
   a) Draw a diagram of this situation, then compare it with a classmate's.
   b) How long are the wires, and how tall is the pole? Express your answers to the nearest tenth of a metre.
11. In $\triangle PQR$, $\angle Q = 90°$, $r = 6$, and $p = 8$. Explain two different ways to determine the measure of $\angle P$. Share your explanation with a partner. How might you improve your explanation?
12. Stella decided to ski to a friend's cabin. She skied 10.0 km in the direction N40°E. She rested, then skied S45°E and arrived at the cabin. The cabin is 14.5 km from her home, as the crow flies. Determine, to the nearest tenth of a kilometre, the distance she travelled on the second leg of her trip.

13. A bridge has been built across a gorge. Jordan wants to bungee jump from the bridge. One of the things she must know, to make the jump safely, is the depth of the gorge. She measured the gorge as shown. Determine the depth of the gorge to the nearest tenth of a metre.

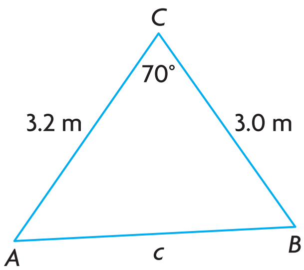


14. Sketch an acute triangle.

a) List three pieces of information about the triangle's sides and angles that would allow you to solve for all the other side lengths and angle measures of the triangle.

b) List three pieces of information about the triangle's sides and angles that would not allow you to solve the triangle.

15. Jim says that the sine law cannot be used to determine the length of side c in $\triangle ABC$. Do you agree or disagree? Explain.



Closing

16. Suppose that you know the length of side p in $\triangle PQR$, as well as the measures of $\angle P$ and $\angle Q$. What other sides and angles could you determine? Explain to a classmate how you would determine these measurements.

Extending

17. In $\triangle ABC$, $\angle A = 58°$, $\angle C = 74°$, and $b = 6$. Determine the area of $\triangle ABC$ to one decimal place.

18. An isosceles triangle has two sides that are 10.0 cm long and two angles that measure 50°. A line segment bisects one of the 50° angles and ends at the opposite side. Determine the length of the line segment to the nearest tenth of a centimetre.

19. Use the sine law to write a ratio that is equivalent to each expression for $\triangle ABC$.

a) $\dfrac{\sin A}{\sin B}$ **b)** $\dfrac{a}{c}$ **c)** $\dfrac{a \sin C}{c \sin A}$