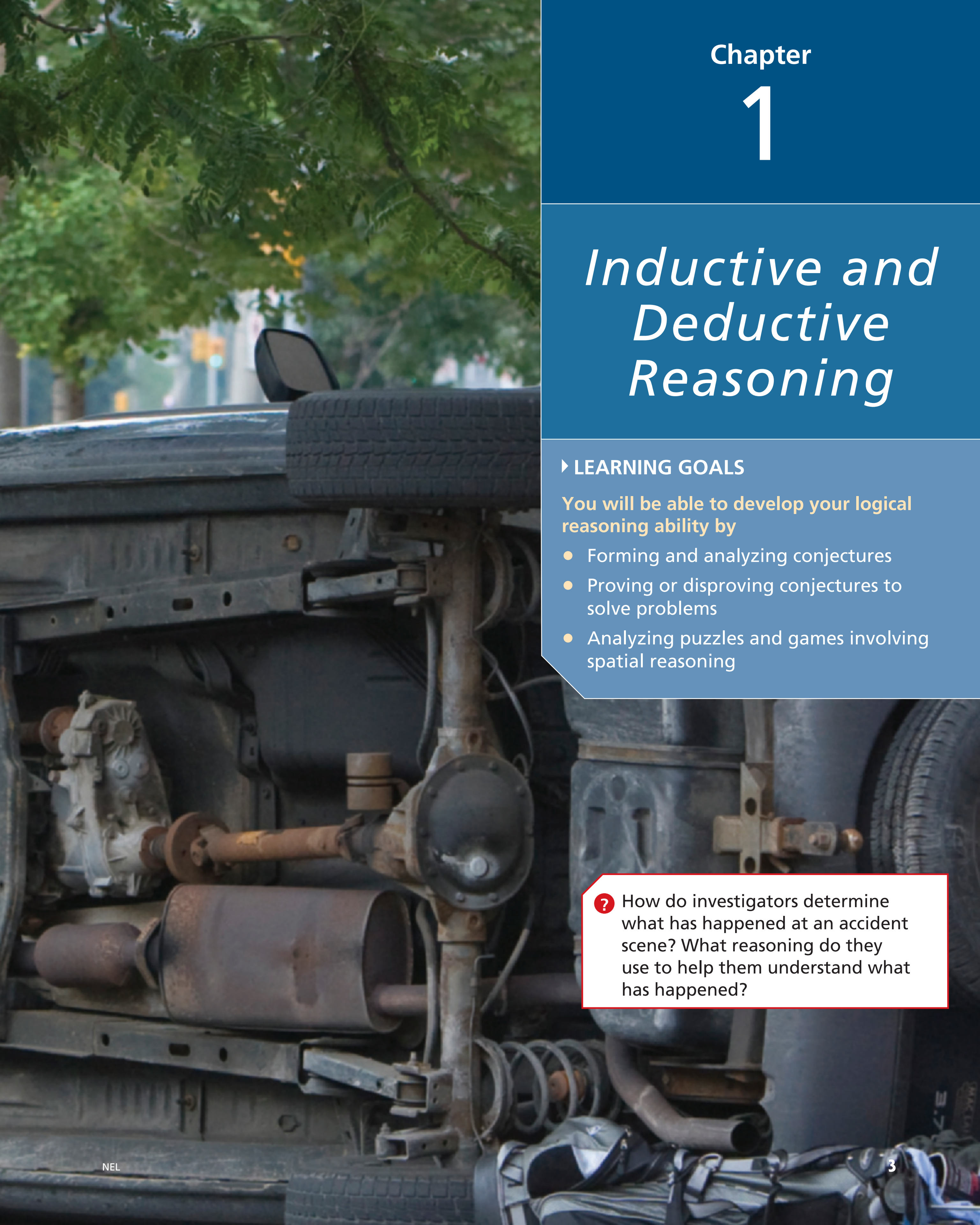


Chapter

1

Inductive and Deductive Reasoning

LEARNING GOALS

You will be able to develop your logical reasoning ability by

- Forming and analyzing conjectures
- Proving or disproving conjectures to solve problems
- Analyzing puzzles and games involving spatial reasoning

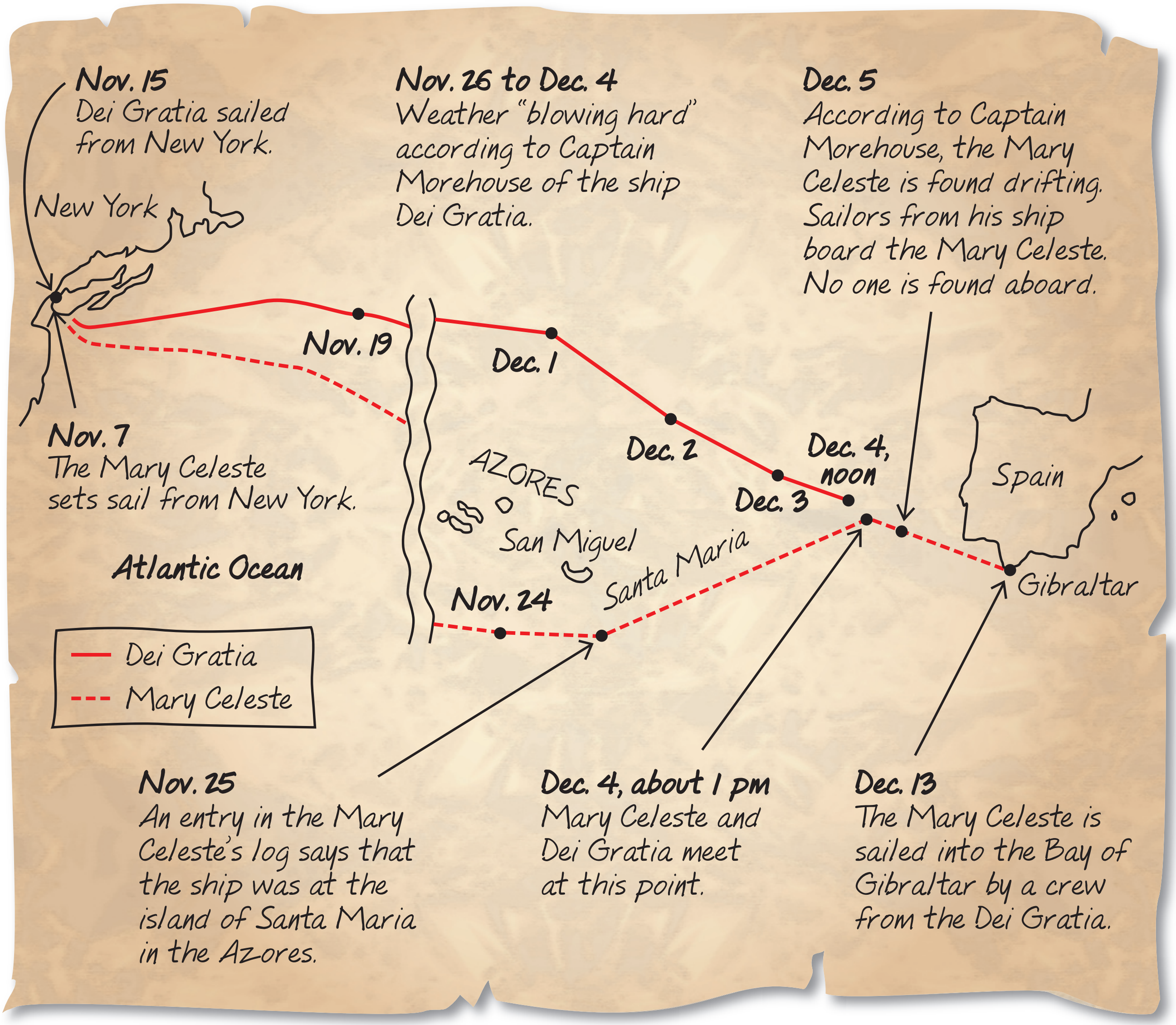
? How do investigators determine what has happened at an accident scene? What reasoning do they use to help them understand what has happened?

1 Getting Started

The Mystery of the *Mary Celeste*

The *Mary Celeste*, a Canadian-built 100-foot brigantine, set sail from New York on November 7, 1872, with Captain Briggs, his wife Sarah, his two-year old daughter Sophia, and a crew of seven. Five weeks later, on December 13, 1872, the *Mary Celeste* sailed into the Bay of Gibraltar with a completely different crew.

Timeline for the Voyage of the *Mary Celeste*



Condition of the *Mary Celeste*, as Reported to Captain Morehouse

- The ship's hull was not damaged.
- No crew or passengers were on board.
- No boats were on board.
- Ropes were dangling over the sides of the ship.
- Only one of two pumps was working.
- The forward and stern hatches were open.
- Water was found between the decks.
- The only dry clothing was found in a watertight chest.
- Kitchenware was scattered and loose in the galley.
- The galley stove was out of place.
- No chronometer or sextant was found on board. Both of these instruments are used for navigation.
- The ship's clock and compass were not working.
- The ship's register was missing. The ship's register is a document that notes home port and country of registration.
- The ship's papers were missing. These papers could have included a bill of sale, ownership information, crew manifest, and cargo information.
- The cargo, 1701 barrels of commercial alcohol, had not shifted. When unloaded in Genoa, 9 barrels were found to be empty.
- The alcohol was not safe to drink, but it could have been burned.

The ship shown here is the *Niagara*, which was built as a tribute to ships of the past, but it is of the same class as the *Mary Celeste*. Under full sail, the brigantine *Mary Celeste* would have looked a lot like this.

? How can you use this information to develop a plausible explanation about what happened to the crew?

A. With a partner, decide what pieces of evidence are most significant.

B. Discuss with your partner possible explanations for the evidence.

C. Choose one explanation, and develop an argument to support it.

D. Find a pair of students with a different explanation. Share your ideas.

E. Build a consensus for an explanation that your group of four could support.

F. What is one other piece of evidence, currently missing, that would further support your explanation?

WHAT DO You Think?

Decide whether you agree or disagree with each statement. Explain your decision.

1. When studying evidence or examples, the patterns you see will lead you to the correct conclusion.
2. Examining examples can help you discover patterns. A pattern can be useful for making predictions.
3. There is only one pattern that can be used to predict the next three terms after 1, 4, 9, 16, 25, ….

1.1

Making Conjectures: Inductive Reasoning

YOU WILL NEED

- calculator
- compass, protractor, and ruler, or dynamic geometry software

EXPLORE…

- If the first three colours in a sequence are red, orange, and yellow, what colours might be found in the rest of the sequence? Explain.

conjecture

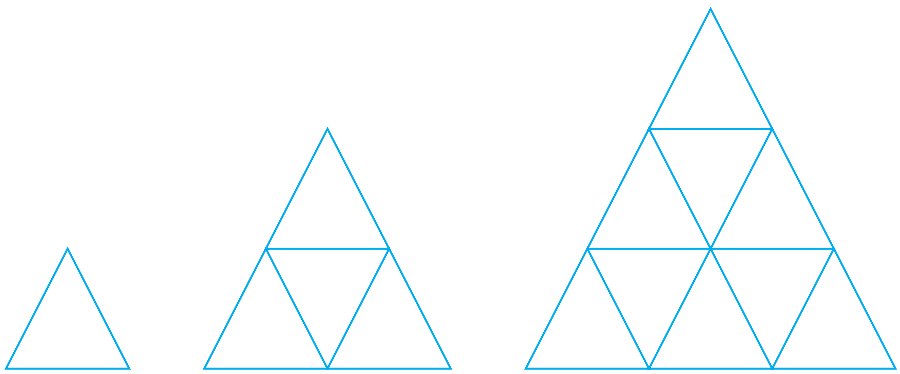
A testable expression that is based on available evidence but is not yet proved.

GOAL

Use reasoning to make predictions.

INVESTIGATE the Math

Georgia, a fabric artist, has been patterning with equilateral triangles. Consider Georgia's **conjecture** about the following pattern.



I think Figure 10 in this pattern will have 100 triangles, and all these triangles will be congruent to the triangle in Figure 1.

? How did Georgia arrive at this conjecture?

A. Organize the information about the pattern in a table like the one below.

Figure	1	2	3
Number of Triangles	1	4	

B. With a partner, discuss what you notice about the data in the table.

C. Extend the pattern for two more figures.

D. What numeric pattern do you see in the table?

Reflecting

E. Is Georgia's conjecture reasonable? Explain.

F. How did Georgia use **inductive reasoning** to develop her conjecture?

G. Is there a different conjecture you could make based upon the pattern you see? Explain.

inductive reasoning

Drawing a general conclusion by observing patterns and identifying properties in specific examples.

APPLY the Math

EXAMPLE 1 Using inductive reasoning to make a conjecture about annual precipitation

Lila studied the following five-year chart for total precipitation in Vancouver.

Precipitation in Vancouver (mm)												
	Jan.	**Feb.**	**Mar.**	**Apr.**	**May**	**Jun.**	**Jul.**	**Aug.**	**Sep.**	**Oct.**	**Nov.**	**Dec.**
2003	150.5	27.1	133.7	139.8	49.3	12.8	19.8	4.1	40.2	248.2	167.4	113.2
2004	249.6	45.8	132.8	90.2	68.6	49.6	43.6	28.6	53.6	155.4	136.6	160.8
2005	283.6	57.0	92.4	70.0	42.8	54.4	25.2	4.8	39.4	57.8	350.8	146.0
2006	181.4	116.0	214.8	76.2	37.0	80.0	53.0	8.4	73.6	155.2	116.2	210.6
2007	137.6	68.6	75.2	62.2	43.2	43.0	15.8	75.8	30.6	99.6	177.0	197.2

Environment Canada, National Climate Data and Information Archive

What conjecture could Lila make based on the data?

Lila's Solution

	Jul.	**Aug.**	**Sep.**
	19.8	4.1	40.2
	43.6	28.6	53.6
	25.2	4.8	39.4
	53.0	8.4	73.6
	15.8	75.8	30.6
Totals:	157.4	121.7	237.4

I looked for patterns in the data. I noticed that the summer months seemed to have less precipitation than the other months. I checked the sum of the precipitation in July, August, and September over the five-year period.

	Jan.	**Feb.**	**Mar.**
	150.5	27.1	133.7
	249.6	45.8	132.8
	283.6	57.0	92.4
	181.4	116.0	214.8
	137.6	68.6	75.2
Totals:	1002.7	314.5	648.9

Then I looked for the months with the greatest precipitation, anticipating that the winter months might have greater precipitation. I checked the sums for January, February, and March.

	Nov.
	167.4
	136.6
	350.8
	116.2
	177.0
Total:	948.0

When I examined the information further, I saw that November had the highest value for precipitation: 350.8 mm. I checked the sum for November.

My conjecture is that fall and winter have more precipitation than spring and summer.

Apr.	May	Jun.	Jul.	Aug.	Sep.
438.4	240.9	239.8	157.4	121.7	237.4

Total: 1435.6 mm

Oct.	Nov.	Dec.	Jan.	Feb.	Mar.
716.2	948.0	827.8	1002.7	314.5	648.9

Total: 4458.1 mm

The data support my conjecture.

Since November is in the fall and January, February, and most of March are in the winter, I can make a conjecture about which seasons have the most precipitation.

I checked the totals for the five-year period. I found that spring and summer had a total of 1435.6 mm of precipitation, and fall and winter had a total of 4458.1 mm of precipitation.

Your Turn

Make a different conjecture based on patterns in the precipitation chart.

EXAMPLE 2 Using inductive reasoning to develop a conjecture about integers

Make a conjecture about the product of two odd integers.

Jay's Solution

$(+3)(+7) = (+21)$

Odd integers can be negative or positive. I tried two positive odd integers first. The product was positive and odd.

$(-5)(-3) = (+15)$

Next, I tried two negative odd integers. The product was again positive and odd.

$(+3)(-3) = (-9)$

Then I tried the other possible combination: one positive odd integer and one negative odd integer. This product was negative and odd.

My conjecture is that the product of two odd integers is an odd integer.

I noticed that each pair of integers I tried resulted in an odd product.

$(-211)(-17) = (+3587)$

I tried other integers to test my conjecture. The product was again odd.

Your Turn

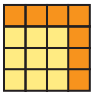
Do you find Jay's conjecture convincing? Why or why not?

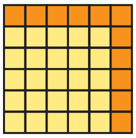
EXAMPLE 3 Using inductive reasoning to develop a conjecture about perfect squares

Make a conjecture about the difference between consecutive perfect squares.

Steffan's Solution: Comparing the squares geometrically

I represented the difference using unit tiles for each perfect square. First, I made a 3×3 square in orange and placed a yellow 2×2 square on top. When I subtracted the 2×2 square, I had 5 orange unit tiles left.

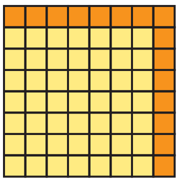




Next, I made 3×3 and 4×4 squares. When I subtracted the 3×3 square, I was left with 7 orange unit tiles. I decided to try greater squares.

My conjecture is that the difference between consecutive squares is always an odd number.

I saw the same pattern in all my examples: an even number of orange unit tiles bordering the yellow square, with one orange unit tile in the top right corner. So, there would always be an odd number of orange unit tiles left, since an even number plus one is always an odd number.



I tested my conjecture with the perfect squares 7×7 and 8×8. The difference was an odd number.

The example supports my conjecture.

Francesca's Solution: Describing the difference numerically

$$2^2 - 1^2 = 4 - 1$$
$$2^2 - 1^2 = 3$$

I started with the smallest possible perfect square and the next greater perfect square: 1^2 and 2^2. The difference was 3.

$$4^2 - 3^2 = 7$$
$$9^2 - 8^2 = 17$$

Then I used the perfect squares 3^2 and 4^2. The difference was 7. So, I decided to try even greater squares.

My conjecture is that the difference between consecutive perfect squares is always a prime number.

I thought about what all three differences—3, 7, and 17—had in common. They were all prime numbers.

$$12^2 - 11^2 = 23$$

To test my conjecture, I tried the perfect squares 11^2 and 12^2. The difference was a prime number.

The example supports my conjecture.

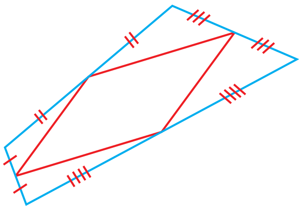
Your Turn

How is it possible to have two different conjectures about the same situation? Explain.

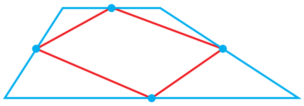
EXAMPLE 4 Using inductive reasoning to develop a conjecture about quadrilaterals

Make a conjecture about the shape that is created by joining the midpoints of adjacent sides in any quadrilateral.

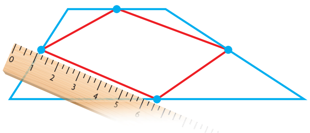
Marc's Solution: Using a protractor and ruler



I drew an irregular quadrilateral on tracing paper. I used my ruler to determine the midpoints of each side. I joined the midpoints of adjacent sides to form a new quadrilateral. This quadrilateral looked like a parallelogram.



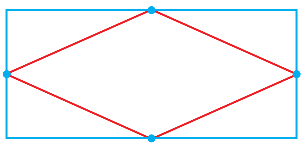
Next, I drew a trapezoid with sides that were four different lengths. I determined the midpoints of the sides. When the midpoints were joined, the new quadrilateral looked like a parallelogram.



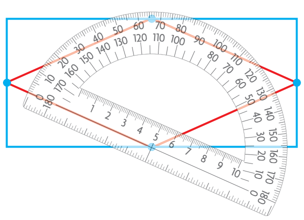
I used my ruler to confirm that the opposite sides were equal.

My conjecture is that joining the adjacent midpoints of any quadrilateral will create a parallelogram.

Each time I joined the midpoints, a parallelogram was formed.



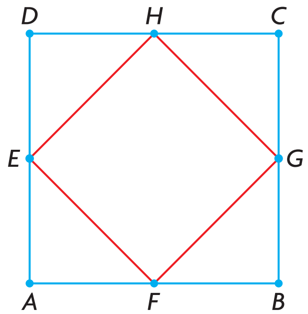
To check my conjecture one more time, I drew a rectangle. I determined its midpoints and joined them. This quadrilateral also looked like a parallelogram.



I checked the measures of the angles in the new quadrilateral. The opposite angles were equal. The new quadrilateral was a parallelogram, just like the others were.

The rectangle example supports my conjecture.

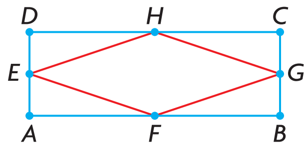
Tracey's Solution: Using dynamic geometry software



$\overline{HE} = 2.4$ cm
$\overline{EF} = 2.4$ cm
$\overline{FG} = 2.4$ cm
$\overline{GH} = 2.4$ cm

$\angle EFG = 90°$
$\angle FGH = 90°$
$\angle GHE = 90°$
$\angle HEF = 90°$

I constructed a square and the midpoints of the sides. Then I joined the adjacent midpoints. *EFGH* looked like a square. I checked its side lengths and angle measures to confirm that it was a square.



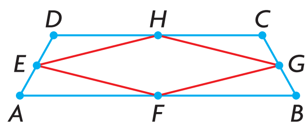
$\overline{HE} = 1.6$ cm
$\overline{EF} = 1.6$ cm
$\overline{FG} = 1.6$ cm
$\overline{GH} = 1.6$ cm

$\angle EFG = 143°$
$\angle FGH = 37°$
$\angle GHE = 143°$
$\angle HEF = 37°$

Next, I constructed a rectangle and joined the adjacent midpoints to create a new quadrilateral, *EFGH*. The side lengths and angle measures of *EFGH* showed that *EFGH* was a rhombus but not a square.

My conjecture is that the quadrilateral formed by joining the adjacent midpoints of any quadrilateral is a rhombus.

Since a square is a rhombus with right angles, both of my examples resulted in a rhombus.



$\overline{HE} = 1.6$ cm
$\overline{EF} = 1.6$ cm
$\overline{FG} = 1.6$ cm
$\overline{GH} = 1.6$ cm

$\angle EFG = 152°$
$\angle FGH = 28°$
$\angle GHE = 152°$
$\angle HEF = 28°$

To check my conjecture, I tried an isosceles trapezoid. The new quadrilateral, *EFGH*, was a rhombus.

The isosceles trapezoid example supports my conjecture.

Your Turn

a) Why did the students draw different conjectures?

b) Do you think that both conjectures are valid? Explain.

In Summary

Key Idea

- Inductive reasoning involves looking at specific examples. By observing patterns and identifying properties in these examples, you may be able to make a general conclusion, which you can state as a conjecture.

Need to Know

- A conjecture is based on evidence you have gathered.
- More support for a conjecture strengthens the conjecture, but does not prove it.

CHECK Your Understanding

Canada's most popular ski destination is Whistler/ Blackcomb in British Columbia. This area draws more than 2 million visitors each year.

1. Troy works at a ski shop in Whistler, British Columbia, where three types of downhill skis are available: parabolic, twin tip, and powder. The manager of the store has ordered 100 pairs of each type, in various lengths, for the upcoming ski season. What conjecture did the manager make? Explain.

2. Tomas gathered the following evidence and noticed a pattern.

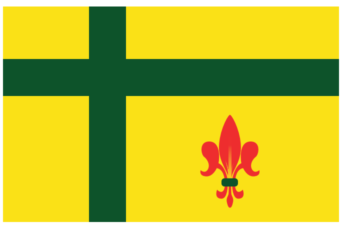
$17(11) = \mathbf{187} \qquad 23(11) = \mathbf{253}$

$41(11) = \mathbf{451} \qquad 62(11) = \mathbf{682}$

Tomas made this conjecture: When you multiply a two-digit number by 11, the first and last digits of the product are the digits of the original number. Is Tomas's conjecture reasonable? Develop evidence to test his conjecture and determine whether it is reasonable.

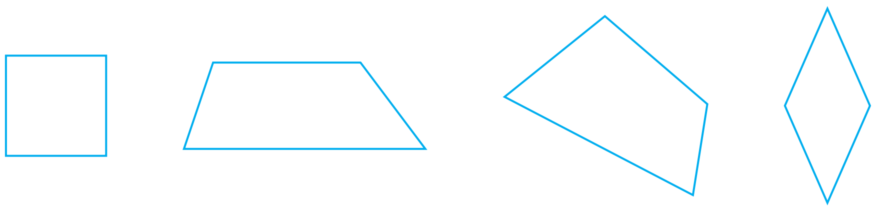
PRACTISING

3. Make a conjecture about the sum of two even integers. Develop evidence to test your conjecture.

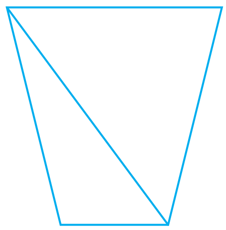
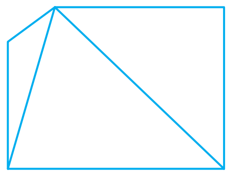
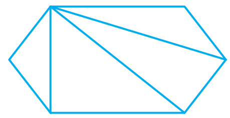


4. Make a conjecture about the meaning of the symbols and colours in the Fransaskois flag, at the left. Consider French-Canadian history and the province of Saskatchewan.

5. Marie studied the sum of the angles in quadrilaterals and made a conjecture. What conjecture could she have made?



6. Use the evidence given in the chart below to make a conjecture. Provide more evidence to support your conjecture.

Polygon	quadrilateral	pentagon	hexagon
Fewest Number of Triangles	 2	 3	 4

7. Sonia noticed a pattern when dividing the square of an odd number by 4. Determine the pattern and make a conjecture.

8. Dan noticed a pattern in the digits of the multiples of 3. He created the following table to show the pattern.

Multiples of 3	12	15	18	21	24	27	30
Sum of the Digits	3	6	9	3	6	9	3

a) Make a conjecture based on the pattern in the table.
b) Find a classmate who made a different conjecture. Discuss the reasonableness of both conjectures.
c) Test one of the conjectures.

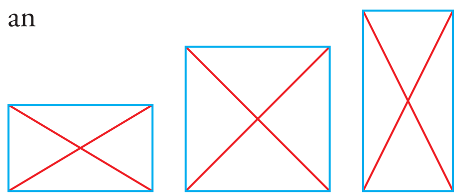
9. Make a conjecture about the sum of one odd integer and one even integer. Test your conjecture with at least three examples.

10. Make a conjecture about the temperature on November 1 in Hay River, Northwest Territories, based on the information in the chart below. Summarize the evidence that supports your conjecture.

Year	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009
Maximum Temperature (°C)	+3.1	−2.2	−1.1	−10.1	−1.6	−3.9	−3.2	+2.9	+1.8	−3.0

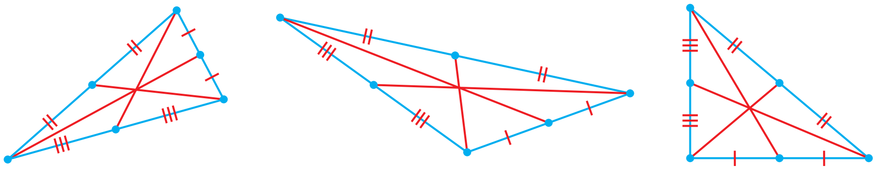
11. Paula claims that whenever you square an odd integer, the result is an odd number. Is her conjecture reasonable? Justify your decision.

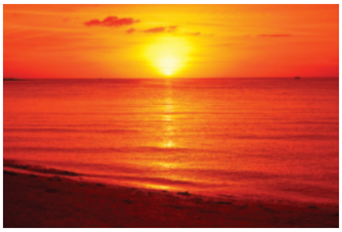
12. Ursula studied the diagonals of these rectangles to look for patterns. Make a conjecture about the diagonals of rectangles. What evidence supports your conjecture?



13. Text messages often include cryptic abbreviations, such as L2G (love to go), 2MI (too much information), LOL (laugh out loud), and MTF (more to follow). Make a conjecture about the cryptic abbreviations used in text messages, and provide evidence to support your conjecture.

14. Nick made a conjecture about the medians of a triangle. He used triangles of different sizes and types to gather evidence. The evidence always supported his conjecture. What might his conjecture have been? Provide additional evidence to support the conjecture.





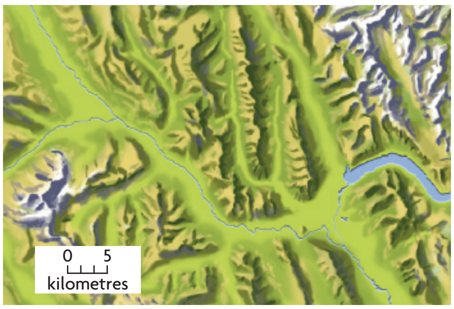
Red sky at sunset may predict calm weather. Based on the photo, how else could you predict that a storm isn't coming?

15. Farmers, travellers, and hunters depend on their observations of weather and storm systems to make quick decisions and to survive in different weather conditions. Weather predictions, passed on through oral tradition or cited in almanacs, are often based on long-term observations. Two predictive statements about weather are given below.

- If cows are lying down, then it is going to rain.
- Red sky at night; sailor's delight.

Find another such predictive statement from oral tradition, an Elder, a family member, an Internet source, or a text. Explain how and why this prediction may have been reached.

16. The map below does not have any roads or urban areas marked. However, there is a town of 7000 people somewhere in this area. Make a conjecture about where the town is. Justify your decision.



17. Suppose that social networking sites were the only method for passing information among people, and that everyone in Canada was a member of one of these sites. Make a conjecture about the amount of time it would take for the entire population of Canada to get information first shared at 8 a.m. Central Standard Time. With a partner, discuss the reasonableness of each of your conjectures and decide how you could gather evidence to support your conjectures.

18. Thérèse held up a piece of notebook paper in one hand and a pair of scissors in the other hand, and made the conjecture that she could walk through the piece of paper. With a partner, explore how Thérèse's conjecture could be possible.

Closing

19. Lou says that conjectures are like inferences in literature and hypotheses in science. Sasha says that conjectures are related only to reasoning. With a partner, discuss these two opinions. Explain how both may be valid.

Extending

20. Photographs lead to conjectures about what was happening around the time that the images were captured. With a partner, develop at least three different conjectures about what could have led to the situation in this photograph.

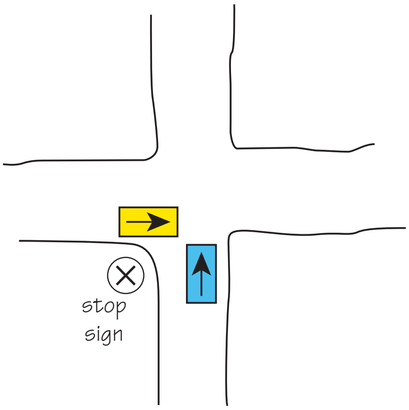
21. In advertising, we often see statements such as "four out of five dentists recommend it." Discuss this statement with a partner, and decide whether it is a conjecture. Justify your decision.

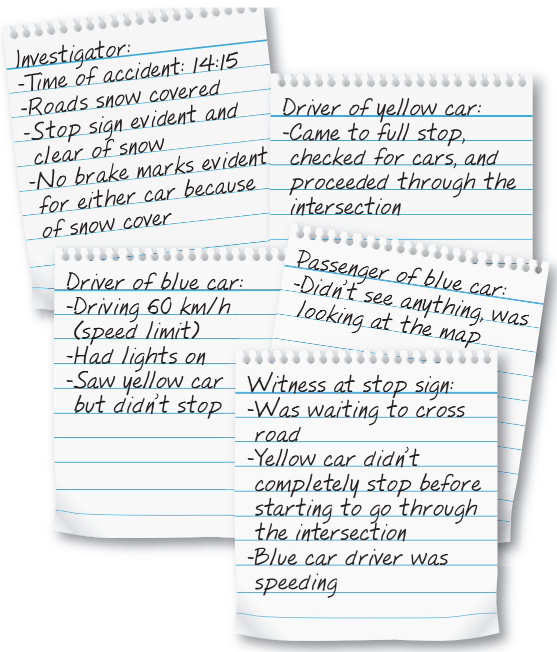
22. From a news source, retrieve evidence about an athlete's current performance or a team's current performance. Study the evidence, and make a conjecture about the athlete's or team's performance over the next month. Justify your conjecture, and discuss the complexity of conjectures about sports.

Math in Action

Oops! What Happened?

- Identify the pieces of given evidence that are conjectures.
- Make a conjecture about what caused the accident.
- What evidence supports your conjecture?
- If you could ask three questions of the drivers or the witness, what would they be?
- Can the cause of an accident such as this be proved? Why or why not?





1.2 Exploring the Validity of Conjectures

YOU WILL NEED
- ruler
- calculator

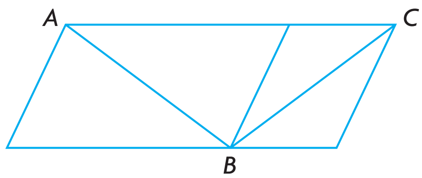
GOAL

Determine whether a conjecture is valid.

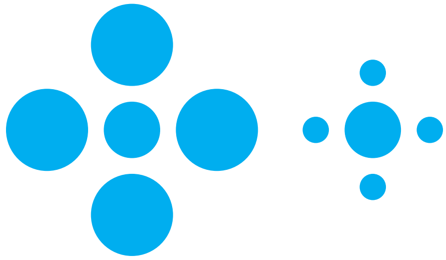
EXPLORE the Math

Your brain can be deceived.

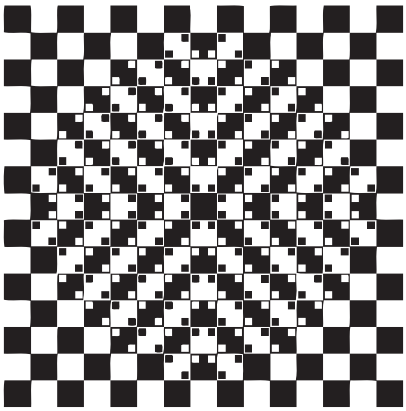
Choose two of these four optical illusions.



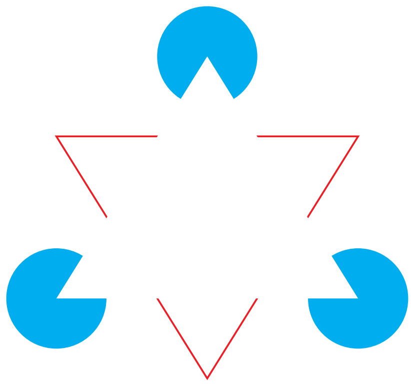
Make a conjecture about diagonal *AB* and diagonal *BC*.



Make a conjecture about the circles in the centre.



Make a conjecture about the lines.



Make a conjecture about the number of triangles.

? How can you check the validity of your conjectures?

Reflecting

A. Describe the steps you took to verify your conjectures.

B. After collecting evidence, did you decide to revise either of your conjectures? Explain.

C. Can you be certain that the evidence you collect leads to a correct conjecture? Explain.

In Summary

Key Idea

- Some conjectures initially seem to be valid, but are shown not to be valid after more evidence is gathered.

Need to Know

- The best we can say about a conjecture reached through inductive reasoning is that there is evidence either to support or deny it.
- A conjecture may be revised, based on new evidence.

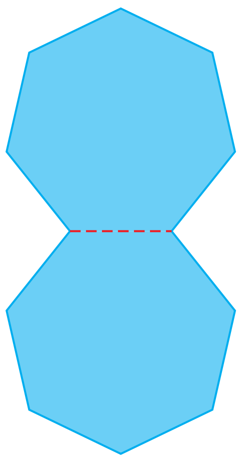
FURTHER Your Understanding

1. Make a conjecture about the dimensions of the two tabletops. How can you determine if your conjecture is valid?

2. Examine the number pattern. Make a conjecture about this pattern. What steps can you take to determine if your conjecture is valid?

$$1^2 = 1$$
$$11^2 = 121$$
$$111^2 = 12321$$
$$1111^2 = 1234321$$

3. If two congruent regular heptagons are positioned so that they share a side, a dodecagon (12-sided polygon) is formed. If two congruent regular hexagons are positioned so that they share a side, a decagon is formed. If two congruent regular pentagons are positioned so that they share a side, an octagon is formed. Make a conjecture about positioning two congruent regular quadrilaterals so that they share a side. Determine whether your conjecture is valid. Record your evidence.



1.3 Using Reasoning to Find a Counterexample to a Conjecture

YOU WILL NEED
- calculator
- ruler
- compass

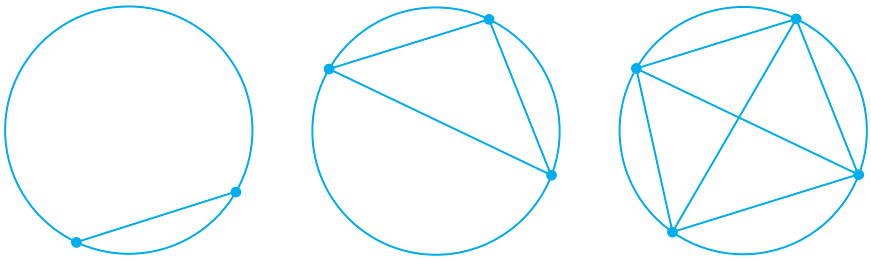
EXPLORE...
- Six, twelve, ten, one, fifty … Conjecture: All but one of the vowels (a, e, i, o, u, and y) are used to spell numbers. Gather evidence to support or deny this conjecture.

GOAL

Invalidate a conjecture by finding a contradiction.

LEARN ABOUT the Math

Kerry created a series of circles. Each circle had points marked on its circumference and joined by chords.



As the number of points on the circumference increased, Kerry noticed a pattern for the number of regions created by the chords.

Number of Points	2	3	4
Number of Regions	2	4	8

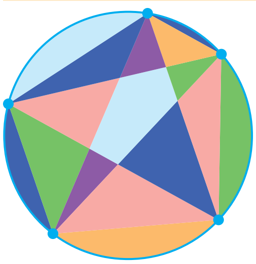
She made the following conjecture: As the number of connected points on the circumference of a circle increases by 1, the number of regions created within the circle increases by a factor of 2.

? How can Kerry test the validity of her conjecture?

EXAMPLE 1 Testing a conjecture

Gather more evidence to test Kerry's conjecture.

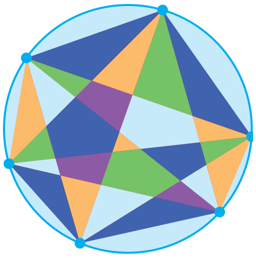
Zohal's Solution



Number of Points	2	3	4	5
Number of Regions	2	4	8	16

I drew another circle and identified five points on its circumference. Then I joined the pairs of points with chords. I coloured the resulting regions to make them easier to count.

My diagram had 16 regions. This supported Kerry's conjecture because the pattern for the resulting regions was 2^1, 2^2, 2^3, 2^4.



Number of Points	2	3	4	5	6
Number of Regions	2	4	8	16	31

I drew another circle and identified six points on its circumference. Then I joined the pairs of points with chords and coloured the regions.

When I counted, I got only 31 regions, not 2^5 or 32 as Kerry's conjecture predicts.

The number of regions did not increase by a factor of 2. This **counterexample** disproves Kerry's conjecture.

counterexample
An example that invalidates a conjecture.

Reflecting

A. Why do you think Zohal started her development of further evidence by using five points on the circumference of a circle?

B. Why is only one counterexample enough to disprove a conjecture?

APPLY the Math

EXAMPLE 2 Connecting to previous conjectures

In Lesson 1.1, page 9, Francesca and Steffan made conjectures about the difference between consecutive squares.

Steffan's conjecture: The difference between consecutive perfect squares is always an odd number.

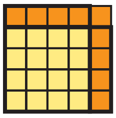
Francesca's conjecture: The difference between consecutive perfect squares is always a prime number.

How can these conjectures be tested?

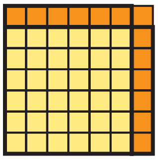
Luke's Solution: Communicating about Steffan's conjecture and more trials

Steffan's conjecture was true for the pairs of consecutive squares he chose: 2×2 and 3×3, 3×3 and 4×4, and 5×5 and 6×6.

First, I tried 1×1 and 2×2. I made the same tile squares as Steffan. When I took away the yellow square, I was left with a pair of tiles that shared an edge with the yellow square and a single tile in the top right corner.

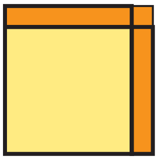


Next, I chose 4 × 4 and 5 × 5, since Steffan had skipped over these values. I was left with two groups of tiles, each with the same value as a side of the yellow square, plus one extra tile in the top right corner.



I tried consecutive squares of 6 × 6 and 7 × 7. The difference again showed the same pattern: two groups of tiles, each with the same value as a side of the yellow square, plus a single tile in the top right corner.

These three examples support Steffan's conjecture.



I visualized what the difference would look like for any pair of consecutive squares. There would always be two groups of orange tiles, each with the same value as a side of the smaller yellow square, plus one unpaired orange square in the corner. The total value of the two equal groups would always be an even number, since 2 times any number is even. The unpaired tile would make the difference odd.

All this evidence strengthens the validity of Steffan's conjecture. However, it doesn't prove the conjecture since I haven't tried all the possible cases.

Pierre's Solution: Connecting more evidence to Francesca's conjecture

Francesca used the consecutive squares of 1 and 2, 3 and 4, and 8 and 9.

I chose values so I could start to fill the gaps between the values that Francesca chose.

$3^2 - 2^2 = 5$

Five is a prime number.

$5^2 - 4^2 = 9$

The next gap was 4 and 5. Nine is not a prime number.

Francesca's conjecture, that the difference between consecutive squares is always a prime number, was disproved since a counterexample was found.

Your Turn

a) Find another counterexample to Francesca's conjecture.

b) Can you find a counterexample to Steffan's conjecture? Explain.

EXAMPLE 3 Using reasoning to find a counterexample to a conjecture

Matt found an interesting numeric pattern:

$$1 \cdot 8 + 1 = \mathbf{9}$$
$$12 \cdot 8 + 2 = \mathbf{98}$$
$$123 \cdot 8 + 3 = \mathbf{987}$$
$$1234 \cdot 8 + 4 = \mathbf{9876}$$

Matt thinks that this pattern will continue.

Search for a counterexample to Matt's conjecture.

Kublu's Solution

$$\mathbf{1} \cdot 8 + \mathbf{1} = 9$$
$$1\mathbf{2} \cdot 8 + \mathbf{2} = 98$$
$$12\mathbf{3} \cdot 8 + \mathbf{3} = 987$$
$$123\mathbf{4} \cdot 8 + \mathbf{4} = 9876$$

The pattern seemed to be related to the first factor (the factor that wasn't 8), the number that was added, and the product.

	A	B
1	1 · 8 + 1	9
2	12 · 8 + 2	98
3	123 · 8 + 3	987
4	1234 · 8 + 4	9876
5	12345 · 8 + 5	98765
6	123456 · 8 + 6	987654
7	1234567 · 8 + 7	9876543
8	12345678 · 8 + 8	98765432
9	123456789 · 8 + 9	987654321

I used a spreadsheet to see if the pattern continued.

The spreadsheet showed that it did.

$$123456789\mathbf{10} \cdot 8 + \mathbf{10} = 98\ 765\ 431\ 290$$
$$123456789\mathbf{0} \cdot 8 + \mathbf{10} = 9\ 876\ 543\ 130$$
$$123456789\mathbf{10} \cdot 8 + \mathbf{0} = 98\ 765\ 431\ 280$$
$$123456789\mathbf{0} \cdot 8 + \mathbf{0} = 9\ 876\ 543\ 120$$

When I came to the tenth step in the sequence, I had to decide whether to use 10 or 0 in the first factor and as the number to add. I decided to check each way that 10 and 0 could be represented.

The pattern holds true until 9 of the 10 digits are included. At the tenth step in the sequence, a counterexample is found.

Since the pattern did not continue, Matt's conjecture is invalid.

Revised conjecture: When the value of the addend is 1 to 9, the pattern will continue.

I decided to revise Matt's conjecture by limiting it.

Your Turn

If Kublu had not found a counterexample at the tenth step, should she have continued looking? When would it be reasonable to stop gathering evidence if all the examples supported the conjecture? Justify your decision.

In Summary

Key Ideas

- Once you have found a counterexample to a conjecture, you have disproved the conjecture. This means that the conjecture is invalid.
- You may be able to use a counterexample to help you revise a conjecture.

Need to Know

- A single counterexample is enough to disprove a conjecture.
- Even if you cannot find a counterexample, you cannot be certain that there is not one. Any supporting evidence you develop while searching for a counterexample, however, does increase the likelihood that the conjecture is true.

CHECK Your Understanding

1. Show that each statement is false by finding a counterexample.
   a) A number that is not negative is positive.
   b) All prime numbers are odd.
   c) All basketball players are tall.
   d) The height of a triangle lies inside the triangle.
   e) On maps, the north arrow always points up.
   f) The square root of a number is always less than the number.
   g) The sum of two numbers is always greater than the greater of the two numbers.
   h) As you travel north, the climate gets colder.

2. Seth claims that all quadrilaterals with four equal sides are squares. Do you agree or disagree? Justify your decision.

PRACTISING

3. Jim claims that whenever you multiply two whole numbers, the product is greater than either of the two factors. Do you agree or disagree? Justify your decision.

4. Rachelle claims that the sum of a multiple of 3 and a multiple of 6 must be a multiple of 6. Do you agree or disagree? Justify your decision.

5. Hannah examined these multiples of 9: 18, 45, 63, 27, 81, 108, 216. She claimed that the sum of the digits in any multiple of 9 will add to 9. Do you agree or disagree? Justify your decision.

6. Colin made the following conjecture: If a quadrilateral has two opposite angles that are right angles, the quadrilateral is a rectangle. Do you agree or disagree? Justify your decision.

7. Claire noticed that the digits 4, 5, 6, and 7 could be used to express each value from 1 to 5 as shown to the right. She conjectured that these digits could be used to express each value from 1 to 20. Explain, with examples, whether Claire's conjecture is reasonable.

Number	Expression
1	$\frac{7-5}{6-4}$
2	$7-6+5-4$
3	$\frac{6(7-5)}{4}$
4	$7+6-5-4$
5	$5(\sqrt{64}-7)$

8. George noted a pattern that was similar to Matt's pattern in Example 3. George conjectured that the products would follow the pattern of ending with the digit 5 or 0. Gather evidence about George's conjecture. Does your evidence strengthen or disprove George's conjecture? Explain.

$$1 \cdot 4 + 1 = 5$$
$$12 \cdot 4 + 2 = 50$$
$$123 \cdot 4 + 3 = 495$$
$$1234 \cdot 4 + 4 = 4940$$

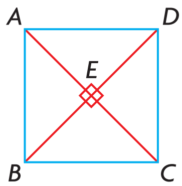
9. From questions 2 to 8, choose a conjecture that you have disproved. Based on your counterexample, revise the conjecture to make it valid.

10. Patrice studied the following table and made this conjecture: The sums of the squares of integers separated by a value of 2 will always be even.

$(-1)^2 + 1^2 = 2$	$2^2 + 4^2 = 20$	$(-3)^2 + (-5)^2 = 34$	$4^2 + 6^2 = 52$	$0^2 + 2^2 = 4$

Is Patrice's conjecture reasonable? Explain.

11. Geoff made the following conjecture: If the diagonals of a quadrilateral are perpendicular, then the quadrilateral is a square. Determine the validity of his conjecture. Explain your results.



12. Amy made the following conjecture: When any number is multiplied by itself, the product will be greater than this starting number. For example, in $2 \cdot 2 = 4$, the product 4 is greater than the starting number 2. Meagan disagreed with Amy's conjecture, however, because $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ and $\frac{1}{4}$ is less than $\frac{1}{2}$. How could Amy's conjecture be improved? Explain the change(s) you would make.

13. Create a general statement that is true in some cases but not in every case. Provide examples that support your statement. Provide a counterexample.

14. Tim conjectured that all natural numbers can be written as the sum of consecutive natural numbers, based on these examples:
$10 = 1 + 2 + 3 + 4$ $\quad$ $12 = 3 + 4 + 5$
$9 = 4 + 5$ $\quad$ $94 = 22 + 23 + 24 + 25$
Do you agree or disagree with Tim's conjecture? Justify your decision.

15. Blake claimed that all odd numbers can be expressed as the sum of three prime numbers. Explain, with evidence, the reasonableness of his claim.

16. German mathematician Christian Goldbach made the conjecture that every even number greater than 2 is the sum of two prime numbers. For example:

$14 = 3 + 11$
$30 = 7 + 23$

This conjecture has become known as Goldbach's conjecture. No one has ever been able to prove that it is true for all even numbers, but no one has ever found a counterexample.

a) Find three other examples that support Goldbach's conjecture.
b) If a counterexample exists, describe what it would look like.

17. Jarrod discovered a number trick in a book he was reading: Choose a number. Double it. Add 6. Double again. Subtract 4. Divide by 4. Subtract 2.

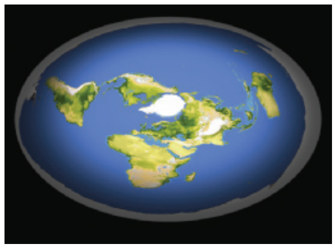
a) Try the trick several times. Make a conjecture about the relation between the number picked and the final result.
b) Can you find a counterexample to your conjecture? What does this imply?

History | Connection

Reasoning in Science

Scientific discoveries are often based on inductive reasoning. Scientists make conjectures after examining all the evidence they have. They test their conjectures by conducting experiments in which they compare how the universe actually behaves with how they predict it should behave. If the experiments have the predicted results, then the scientists' conjectures are strengthened. If the results contradict the conjectures, then the scientists use the results to revise their conjectures or to make new conjectures.

Many scientific conjectures have been changed over time as new information has come to light. One such conjecture relates to Earth itself. In ancient times, the world was believed to be flat. The flat world conjecture was held to be true until counterexamples required it to be changed. Aristotle, in about 330 BCE, was one of the first people to conjecture that the world was not a flat disc but a sphere. Pliny the Elder, in the 1st century CE, was able to suggest that the flat world conjecture was no longer valid because of the evidence that had been developed to contradict it. Pliny considered the possibility of an imperfect sphere. Modern evidence, from satellite images and spaceships, has provided no counterexamples to the spheroid theory, so this theory is generally accepted as fact today.



If Earth were a flat disc, it might look like this from space.

However, it actually looks like this.

A. What other conjectures about our universe have been revised after new evidence was gathered?
B. How does inductive reasoning play a part in our beliefs and understanding about our universe?

Closing

18. What relationship exists among inductive reasoning, evidence, and counterexamples?

Extending

19. Serge made the following conjecture: When 3 is subtracted from a perfect square that is greater than 4, the result is always a composite number. For example:

$15^2 - 3 = 222$

222 is a composite number because it is divisible by factors other than 1 and itself. Do you agree with Serge's conjecture? Justify your decision.

20. Environment Canada explains probability of precipitation forecasts as subjective estimates. These forecasts or estimates are actually conjectures based on numerical evidence and regional topography. They are important for people, such as building contractors, farmers, and surveyors, who work outside and for anyone else who is planning outdoor activities. As the time for precipitation comes closer, a forecaster's conjecture is revised to reflect newer data and increased accuracy.

a) Environment Canada's chart below seems to be written for adults who live in a city or its suburbs. Revise the chart so that it is written for you, by including activities that you participate in.

b) Explain your revisions. How did you decide which probabilities of precipitation could affect your activities?

A User's Guide to Probability of Precipitation

0%	No precipitation even though it may be cloudy.
10%	Little likelihood of rain or snow: only 1 chance in 10.
20%	No precipitation is expected.
30%	If you go ahead with your outdoor plans, keep an eye on the weather.
40%	An umbrella is recommended. Make alternate plans for outdoor activities that are susceptible to rain. Not a good day to pave the driveway. Keep your fingers crossed!
50%	It's 50–50 on whether you get precipitation or not.
60%	Want to water your lawn? The odds are favourable that Mother Nature might give you some help.
70%	Consider the effect of precipitation on your plans for outdoor activities. The chance for no precipitation is only 3 in 10!
80%	Rain or snow is likely.
90%	The occurrence of precipitation is a near certainty.
100%	Precipitation is a certainty.

Environment Canada, Probability of Precipitation brochure

21. Mohammed claims that the expression $n^2 + n + 2$ will never generate an odd number for a positive integer value of n. Do you agree or disagree? Justify your decision.

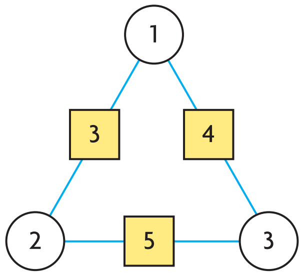
Applying Problem-Solving Strategies

Analyzing a Number Puzzle

Arithmagons are number puzzles that have addition as the central operation. They are based on polygons, with a circle at each vertex and a box on each side.

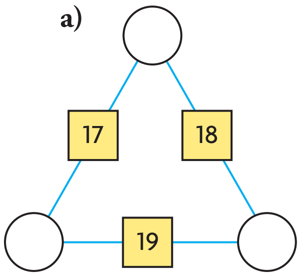
The Puzzle

The number in each box of an arithmagon is the sum of the two numbers in the circles adjacent to the box.

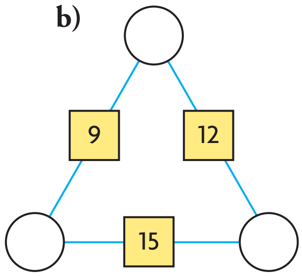


A. Solve the three triangular arithmagons below.

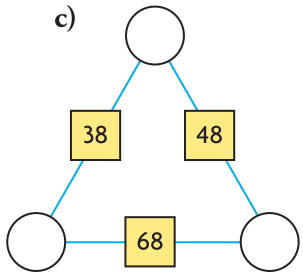
a)



b)



c)



B. Create your own arithmagon. Exchange arithmagons with a partner, and solve your partner's arithmagon.

The Strategy

C. What patterns did you notice?
D. What relationship exists between the numbers in the circles and the numbers in the opposite boxes?
E. Describe the strategy you used to solve your partner's arithmagon.
F. Of the arithmagons you've encountered, which was easiest to solve? Explain.

1.4 Proving Conjectures: Deductive Reasoning

GOAL

Prove mathematical statements using a logical argument.

YOU WILL NEED

- calculator
- ruler

EXPLORE...

- How can the conjecture "All teens like music" be supported inductively? Can this conjecture be proved? Explain.

proof

A mathematical argument showing that a statement is valid in all cases, or that no counterexample exists.

generalization

A principle, statement, or idea that has general application.

LEARN ABOUT the Math

Jon discovered a pattern when adding integers:

$$1 + 2 + 3 + 4 + 5 = 15$$
$$(-15) + (-14) + (-13) + (-12) + (-11) = -65$$
$$(-3) + (-2) + (-1) + 0 + 1 = -5$$

He claims that whenever you add five consecutive integers, the sum is always 5 times the median of the numbers.

? How can you prove that Jon's conjecture is true for all integers?

EXAMPLE 1 Connecting conjectures with reasoning

Prove that Jon's conjecture is true for all integers.

Pat's Solution

$$5(3) = 15$$
$$5(-13) = -65$$
$$5(-1) = -5$$

The median is the middle number in a set of integers when the integers are arranged in consecutive order. I observed that Jon's conjecture was true in each of his examples.

$$210 + 211 + 212 + 213 + 214 = 1060$$
$$5(212) = 1060$$

I tried a sample with greater integers, and the conjecture still worked.

Let x represent any integer.
Let S represent the sum of five consecutive integers.
$S = (x - 2) + (x - 1) + x + (x + 1) + (x + 2)$

I decided to start my **proof** by representing the sum of five consecutive integers. I chose x as the median and then wrote a **generalization** for the sum.

$S = (x + x + x + x + x) + (-2 + (-1) + 0 + 1 + 2)$
$S = 5x + 0$

I simplified by gathering like terms.

$S = 5x$
Jon's conjecture is true for all integers.

Since x represents the median of five consecutive integers, $5x$ will always represent the sum.

deductive reasoning

Drawing a specific conclusion through logical reasoning by starting with general assumptions that are known to be valid.

Reflecting

A. What type of reasoning did Jon use to make his conjecture?

B. Pat used **deductive reasoning** to prove Jon's conjecture. How does this differ from the type of reasoning that Jon used?

APPLY the Math

EXAMPLE 2 Using deductive reasoning to generalize a conjecture

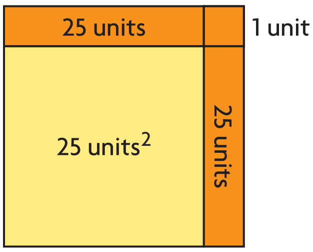
In Lesson 1.3, page 19, Luke found more support for Steffan's conjecture from Lesson 1.1, page 9—that the difference between consecutive perfect squares is always an odd number.

Determine the general case to prove Steffan's conjecture.

Gord's Solution

The difference between consecutive perfect squares is always an odd number.

Steffan's conjecture has worked for consecutive perfect squares with sides of 1 to 7 units.



$$26^2 - 25^2 = 2(25) + 1$$
$$26^2 - 25^2 = 51$$

I tried a sample using even greater squares: 26^2 and 25^2.

The difference is the two sets of 25 unit tiles, plus a single unit tile.

Let x be any natural number.
Let D be the difference between consecutive perfect squares.
$D = (x + 1)^2 - x^2$

Since the conjecture has been supported with specific examples, I decided to express the conjecture as a general statement. I chose x to be the length of the smaller square's sides. The larger square's sides would then be $x + 1$.

$$D = x^2 + x + x + 1 - x^2$$
$$D = x^2 + 2x + 1 - x^2$$
$$D = 2x + 1$$

I expanded and simplified my expression. Since x represents any natural number, $2x$ is an even number, and $2x + 1$ is an odd number.

Steffan's conjecture, that the difference of consecutive perfect squares is always an odd number, has been proved for all natural numbers.

Your Turn

In Lesson 1.3, Luke visualized the generalization but did not develop the reasoning to support it. How did the visualization explained by Luke help Gord develop the general statement? Explain.

EXAMPLE 3 Using deductive reasoning to make a valid conclusion

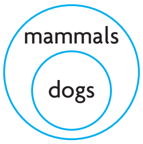
All dogs are mammals. All mammals are vertebrates. Shaggy is a dog.

What can be deduced about Shaggy?

Oscar's Solution

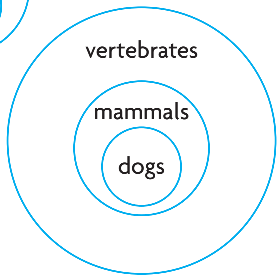
Shaggy is a dog.

All dogs are mammals.



These statements are given. I represented them using a Venn diagram.

All mammals are vertebrates.



This statement is given. I modified my diagram.

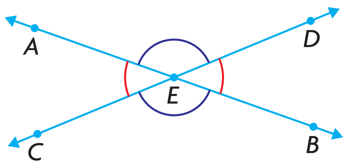
Therefore, through deductive reasoning, Shaggy is a mammal and a vertebrate.

Your Turn

Weight-lifting builds muscle. Muscle makes you strong. Strength improves balance. Inez lifts weights. What can be deduced about Inez?

EXAMPLE 4 Using deductive reasoning to prove a geometric conjecture

Prove that when two straight lines intersect, the vertically opposite angles are equal.



Jose's Solution: Reasoning in a two-column proof

Statement	Justification
$\angle AEC + \angle AED = 180°$	Supplementary angles
$\angle AEC = 180° - \angle AED$	Subtraction property
$\angle BED + \angle AED = 180°$	Supplementary angles
$\angle BED = 180° - \angle AED$	Subtraction property
$\angle AEC = \angle BED$	**Transitive property**

Your Turn

Use a **two-column proof** to prove that $\angle AED$ and $\angle CEB$ are equal.

transitive property

If two quantities are equal to the same quantity, then they are equal to each other.
If $a = b$ and $b = c$, then $a = c$.

two-column proof

A presentation of a logical argument involving deductive reasoning in which the statements of the argument are written in one column and the justifications for the statements are written in the other column.

EXAMPLE 5 Communicating reasoning about a divisibility rule

The following rule can be used to determine whether a number is divisible by 3:

Add the digits, and determine if the sum is divisible by 3. If the sum is divisible by 3, then the original number is divisible by 3.

Use deductive reasoning to prove that the divisibility rule for 3 is valid for two-digit numbers.

Lee's Solution

Expanded Number Forms

Number	Expanded Form (Words)	Expanded Form (Numbers)
9	9 ones	$9(1)$
27	2 tens and 7 ones	$2(10) + 7(1)$
729	7 hundreds and 2 tens and 9 ones	$7(100) + 2(10) + 9(1)$
ab	a tens and b ones	$a(10) + b(1)$

Let ab represent any two-digit number.

I let ab represent any two-digit number.

$ab = 10a + b$

Since any number can be written in expanded form, I wrote ab in expanded form.

$ab = (9a + 1a) + b$
$ab = 9a + (a + b)$

I decomposed $10a$ into an equivalent sum. I used $9a$ because I knew that $9a$ is divisible by 3, since 3 is a factor of 9.

The number ab is divisible by 3 only when $(a + b)$ is divisible by 3.

From this equivalent expression, I concluded that ab is divisible by 3 only when both $9a$ and $(a + b)$ are divisible by 3. I knew that $9a$ is always divisible by 3, so I concluded that ab is divisible by 3 only when $(a + b)$ is divisible by 3.

The divisibility rule has been proved for two-digit numbers.

Your Turn

Use similar reasoning to prove that the divisibility rule for 3 is valid for three-digit numbers.

In Summary

Key Idea

- Deductive reasoning involves starting with general assumptions that are known to be true and, through logical reasoning, arriving at a specific conclusion.

Need to Know

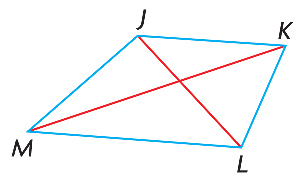
- A conjecture has been proved only when it has been shown to be true for every possible case or example. This is accomplished by creating a proof that involves general cases.
- When you apply the principles of deductive reasoning correctly, you can be sure that the conclusion you draw is valid.
- The transitive property is often useful in deductive reasoning. It can be stated as follows: Things that are equal to the same thing are equal to each other. If $a = b$ and $b = c$, then $a = c$.
- A demonstration using an example is *not* a proof.

CHECK Your Understanding

1. Chuck made the conjecture that the sum of any seven consecutive integers is 7 times the median. Prove Chuck's conjecture.

2. Jim is a barber. Everyone whose hair is cut by Jim gets a good haircut. Austin's hair was cut by Jim. What can you deduce about Austin?

3. Lila drew a quadrilateral and its diagonals. What could Lila deduce about the angles formed at the intersection of the diagonals?

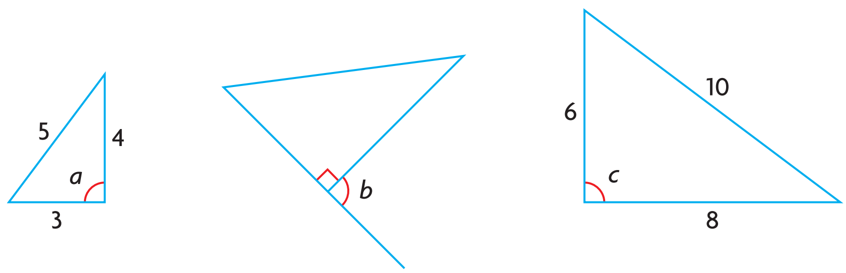


PRACTISING

4. Prove that the sum of two even integers is always even.

5. Prove that the product of an even integer and an odd integer is always even.

6. Prove that a, b, and c are equal.



7. Drew created this step-by-step number trick:
- Choose any number.
- Multiply by 4.
- Add 10.
- Divide by 2.
- Subtract 5.
- Divide by 2.
- Add 3.

a) Show inductively, using three examples, that the result is always 3 more than the chosen number.
b) Prove deductively that the result is always 3 more than the chosen number.

8. Examine the following example of deductive reasoning. Why is it faulty?

Given: Khaki pants are comfortable. Comfortable pants are expensive. Adrian's pants are not khaki pants.
Deduction: Adrian's pants are not expensive.

9. Recall Jarrod's number trick from Lesson 1.3, page 24:
- Choose a number.
- Double it.
- Add 6.
- Double again.
- Subtract 4.
- Divide by 4.
- Subtract 2.

Prove that any number you choose will be the final result.

10. Prove that whenever you square an odd integer, the result is odd.

11. Cleo noticed that whenever she determined the difference between the squares of consecutive even numbers or the difference between the squares of consecutive odd numbers, the result was a multiple of 4. Show inductively that this pattern exists. Then prove deductively that it exists.

12. Create a number trick with five or more steps, similar to the number trick in question 9. Your number trick must always result in a final answer of 6. Prove that your number trick will always work.

13. Prove that any four-digit number is divisible by 2 when the last digit in the number is divisible by 2.

14. Prove that any two-digit or three-digit number is divisible by 5 when the last digit in the number is divisible by 5.

15. To determine if a number is divisible by 9, add all the digits of the number and determine if the sum is divisible by 9. If it is, then the number is divisible by 9. Prove that the divisibility rule for 9 works for all two-digit and three-digit numbers.

16. Look for a pattern when any odd number is squared and then divided by 4. Make a conjecture, and then prove your conjecture.

Closing

17. Simon made the following conjecture: When you add three consecutive numbers, your answer is always a multiple of 3. Joan, Garnet, and Jamie took turns presenting their work to prove Simon's conjecture. Which student had the strongest proof? Explain.

Joan's Work	Garnet's Work	Jamie's Work
$1 + 2 + 3 = 6$ $\quad 3 \cdot 2 = 6$ $2 + 3 + 4 = 9$ $\quad 3 \cdot 3 = 9$ $3 + 4 + 5 = 12$ $\quad 3 \cdot 4 = 12$ $4 + 5 + 6 = 15$ $\quad 3 \cdot 5 = 15$ $5 + 6 + 7 = 18$ $\quad 3 \cdot 6 = 18$ and so on ... Simon's conjecture is valid.	$3 + 4 + 5$ The two outside numbers (3 and 5) add to give twice the middle number (4). All three numbers add to give 3 times the middle number. Simon's conjecture is valid.	Let the numbers be n, $n + 1$, and $n + 2$. $n + n + 1 + n + 2 = 3n + 3$ $n + n + 1 + n + 2 = 3(n + 1)$ Simon's conjecture is valid.

Extending

18. The table below outlines one possible personal strategy for calculating the square of a number.

Step	Method	Example
1	Round the number down to the nearest multiple of 10.	37 is the number to be squared. Round down to 30.
2	Determine the difference between the original number and the rounded number. Add the difference to the original number.	$37 - 30 = 7$ $7 + 37 = 44$
3	Multiply the rounded number by the number from step 2.	$(30)(44) = 1320$
4	Add the square of the difference between the original number and the rounded number.	$1320 + 7^2 = 1369$

From the given example, determine deductively the general rule for x^2.

19. Prove that the expression $n^2 + n + 2$ will always generate an even number for every natural number, n.

20. Make a conjecture about the product of two consecutive natural numbers. Prove your conjecture.

1 Mid-Chapter Review

FREQUENTLY ASKED Questions

Study Aid

- See Lesson 1.1, Examples 1 to 4, and Lesson 1.4, Examples 1 to 5.
- Try Mid-Chapter Review Questions 1 to 3 and 9 to 12.

Q: What is the difference between inductive reasoning and deductive reasoning?

A: Inductive reasoning involves identifying patterns through examples to develop a conjecture, or a general statement. A set of examples, however, is not a proof. The examples can only support the conjecture. In comparison, deductive reasoning proves a general rule, which can then be applied to any specific case.

Inductive Reasoning	Deductive Reasoning
$1 + 2 = 3$ $4 + 5 = 9$ $-7 + (-6) = -13$ $128 + 129 = 257$ From the pattern shown in the examples, I think that the sum of two consecutive integers is always an odd number. But I can't be sure that my conjecture is valid in every case based on only this evidence.	I can represent two consecutive integers as x and $x + 1$. The sum of two consecutive integers can be expressed as $(x) + (x + 1)$. This expression can be simplified to $2x + 1$. Since 2x is an even number, then $2x + 1$ is an odd number. Since the conjecture has been proved, I know that it is valid for any two consecutive integers.

Study Aid

- See Lesson 1.3, Examples 1 to 3.
- Try Mid-Chapter Review Questions 5 to 8.

Q: What is the purpose of finding counterexamples?

A: Counterexamples are examples that disprove a conjecture. Counterexamples may be used to revise the conjecture.

For example,
Conjecture: In November, daily temperatures in Hay River, Northwest Territories, do not exceed 10 °C.
Counterexample: On November 19, 2005, the maximum daily temperature was 11.8 °C.
Revised conjecture: In November, daily temperatures in Hay River, Northwest Territories, do not exceed 10 °C more than once every five years.

PRACTISING

Lesson 1.1

1. A medicine wheel consists of a cairn of stones surrounded by a circle of rocks, with lines of rocks extending from the centre to the circle. The Moose Mountain Medicine Wheel is a sacred site, created by First Nations peoples more than 2000 years ago. Its exact purpose is not known. Make a conjecture about the purpose or usage of the Medicine Wheel.

2. Evaluate the following squares.

67^2 $\quad 667^2 \quad 6667^2 \quad 66667^2$

What pattern do you notice? Make a conjecture about the 25th term in the pattern.

3. Part of Pascal's triangle is shown below. The column on the right represents the sums of the numbers in the rows of Pascal's triangle.

1	1
1 1	2
1 2 1	4
1 3 3 1	8
1 4 6 4 1	16
1 5 10 10 5 1	32
1 6 15 20 15 6 1	64

a) Based on the evidence in the column on the right, make a conjecture about the sum of the numbers in the 10th row.

b) Make a conjecture about the sum of any row.

Lesson 1.2

4. Glenda found a website that had a list of all the countries in the world. As she scanned the list, she made the conjecture that more of the names end with a vowel than a consonant. Gather evidence about Glenda's conjecture. How reasonable is her conjecture?

Lesson 1.3

5. Tony claims that the best hockey players in the world are from Canada. Find a counterexample to his claim.

6. Leanne says that if the diagonals of a quadrilateral bisect each other, then the quadrilateral is a rectangle. Do you agree or disagree? Explain.

7. Ned says that if you can find 10 or more examples that support a mathematical statement, then you can conclude that the statement is always true. Do you agree or disagree? Explain.

Lesson 1.4

8. Use inductive reasoning to make a conjecture about each number trick below. Then use deductive reasoning to prove your conjecture.

a) Choose a number. Add 3. Multiply by 2. Add 4. Divide by 2. Subtract the number you started with. What is the result?

b) Choose a number. Double it. Add 9. Add the number you started with. Divide by 3. Add 4. Subtract the number you started with. What is the result?

9. Prove that the sum of four consecutive natural numbers is always even.

10. Consider the following statement: The square of the sum of two positive integers is greater than the sum of the squares of the same two integers. Test this statement inductively with three examples, and then prove it deductively.

11. Prove that the difference between the square of any odd integer and the integer itself is always an even integer.

1.5 Proofs That Are Not Valid

YOU WILL NEED

- grid paper
- ruler
- scissors

EXPLORE…

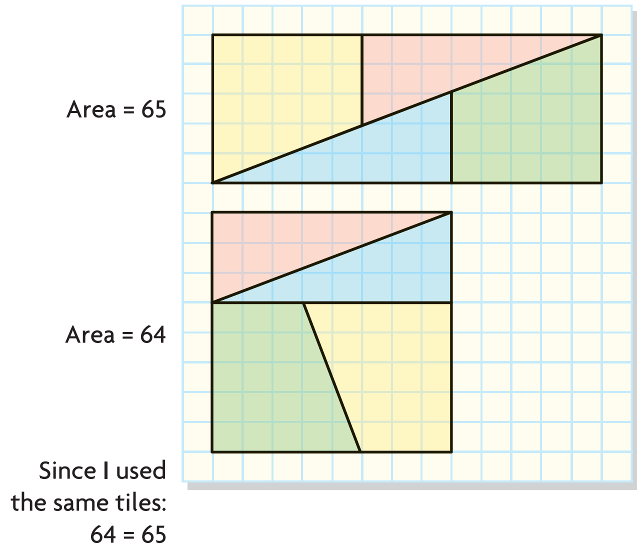
- Consider the following statement: There are tthree errorss in this sentence. Is the statement valid?

GOAL

Identify errors in proofs.

INVESTIGATE the Math

Moh was working with tiles on grid paper. He used right triangles and right trapezoids.



? How was it possible for 64 to equal 65?

A. Construct, as precisely as possible, the square figure on grid paper.

B. Separate the square into its right triangles and right trapezoids.

C. Cut out the shapes. Then reconfigure the shapes to make the rectangle.

D. Determine the accuracy of the positions of the shapes by looking for overlap or empty space.

E. Does Moh's rearrangement of tiles prove that 64 equals 65? Explain.

Reflecting

F. What does any overlap or empty space suggest about the areas of the figures?

G. How do the colours make the rectangle and the square appear to have the same area?

H. Explain how you can check for errors in your constructions.

APPLY the Math

EXAMPLE 1 Using reasoning to determine the validity of an argument

Athletes do not compete in both the Summer and Winter Olympics. Hayley Wickenheiser has represented Canada four times at the Winter Olympics. Therefore, Hayley Wickenheiser has not participated in the Summer Olympics.

Hayley Wickenheiser has represented Canada four times at the Winter Olympics.

Tia read these statements and knew that there was an error. Identify the error in the reasoning.

Tia's Solution

Athletes do not compete in both the Summer and Winter Olympics.

I did some research and found that 18 athletes have competed in both Games. This statement is not valid.

Hayley Wickenheiser has represented Canada four times at the Winter Olympics.

This statement is true. She has played on the national hockey team.

Therefore, Hayley Wickenheiser has not participated in the Summer Olympics.

The conclusion was false because the first statement was false. Hayley played for Canada in the softball competition in the 2000 Summer Olympics.

Your Turn

Zack is a high school student. All high school students dislike cooking. Therefore, Zack dislikes cooking. Where is the error in the reasoning?

Communication *Tip*

Stereotypes are generalizations based on culture, gender, religion, or race. There are always counterexamples to stereotypes, so conclusions based on stereotypes are not valid.

EXAMPLE 2 Using reasoning to determine the validity of a proof

Bev claims he can prove that $3 = 4$.

Bev's Proof

Suppose that: $a + b = c$

This statement can be written as: $4a - 3a + 4b - 3b = 4c - 3c$

After reorganizing, it becomes: $4a + 4b - 4c = 3a + 3b - 3c$

Using the distributive property, $4(a + b - c) = 3(a + b - c)$

Dividing both sides by $(a + b - c)$, $4 = 3$

Show that Bev has written an **invalid proof**.

invalid proof

A proof that contains an error in reasoning or that contains invalid assumptions.

Pru's Solution

Suppose that:

$$a + b = c$$ ✔

Bev's **premise** was made at the beginning of the proof. Since variables can be used to represent any numbers, this part of the proof is valid.

premise

A statement assumed to be true.

$$4a - 3a + 4b - 3b = 4c - 3c$$ ✔

Bev substituted $4a - 3a$ for a since $4a - 3a = a$.
Bev substituted $4b - 3b$ for b since $4b - 3b = b$.
Bev substituted $4c - 3c$ for c since $4c - 3c = c$.

$$4a + 4b - 4c = 3a + 3b - 3c$$ ✔

I reorganized the equation and I came up with the same result that Bev did when he reorganized. Simplifying would take me back to the premise. This part of the proof is valid.

$$4(a + b - c) = 3(a + b - c)$$ ✔

Since each side of the equation has the same coefficient for all the terms, factoring both sides is a valid step.

$$\frac{4(a + b - c)}{(a + b - c)} = \frac{3(a + b - c)}{(a + b - c)}$$

This step appears to be valid, but when I looked at the divisor, I identified the flaw.

$$a + b = c$$
$$a + b - c = c - c$$
$$a + b - c = 0$$

When I rearranged the premise, I determined that the divisor equalled zero.

Dividing both sides of the equation by $a + b - c$ is not valid. Division by zero is undefined.

Your Turn

How could this type of false proof be used to suggest that $65 = 64$?

EXAMPLE 3 Using reasoning to determine the validity of a proof

Liz claims she has proved that $-5 = 5$.

Liz's Proof

I assumed that $-5 = 5$.
Then I squared both sides: $(-5)^2 = 5^2$
I got a true statement: $25 = 25$
This means that my assumption, $-5 = 5$, must be correct.

Where is the error in Liz's proof?

Simon's Solution

I assumed that $-5 = 5$.

Liz started off with the false assumption that the two numbers were equal.

Then I squared both sides: $(-5)^2 = 5^2$
I got a true statement: $25 = 25$

Everything that comes after the false assumption doesn't matter because the reasoning is built on the false assumption.

Even though $25 = 25$, the underlying premise is not true.

$-5 \neq 5$

Liz's conclusion is built on a false assumption, and the conclusion she reaches is the same as her assumption.

If an assumption is not true, then any argument that was built on the assumption is not valid.

Circular reasoning has resulted from these steps. Starting with an error and then ending by saying that the error has been proved is arguing in a circle.

circular reasoning
An argument that is incorrect because it makes use of the conclusion to be proved.

Your Turn

How is an error in a premise like a counterexample?

EXAMPLE 4 Using reasoning to determine the validity of a proof

Hossai is trying to prove the following number trick:
Choose any number. Add 3. Double it. Add 4. Divide by 2. Take away the number you started with.

Each time Hossai tries the trick, she ends up with 5. Her proof, however, does not give the same result.

Hossai's Proof

n	Choose any number.
$n + 3$	Add 3.
$2n + 6$	Double it.
$2n + 10$	Add 4.
$2n + 5$	Divide by 2.
$n + 5$	Take away the number you started with.

Where is the error in Hossai's proof?

Sheri's Solution

$1 \longrightarrow 5$
$10 \longrightarrow 5$

I tried the number trick twice, for the number 1 and the number 10. Both times, I ended up with 5. The math trick worked for Hossai and for me, so the error must be in Hossai's proof.

n ✔

The variable *n* can represent any number. This step is valid.

$n + 3$ ✔

Adding 3 to *n* is correctly represented.

$2n + 6$ ✔

Doubling a quantity is multiplying by 2. This step is valid. Its simplification is correct as well.

$2n + 10$ ✔

Adding 4 to the expression is correctly represented, and the simplification is correct.

$2n + 5$ ✘

The entire expression should be divided by 2, not just the constant. This step is where the mistake occurred.

I corrected the mistake:

$$\frac{2n + 10}{2} = n + 5$$
$$n + 5 - n = 5$$

I completed Hossai's proof by subtracting *n*. I showed that the answer will be 5 for any number.

Your Turn

Is there a number that will not work in Hossai's number trick? Explain.

EXAMPLE 5 Using reasoning to determine the validity of a proof

Jean says she can prove that $\$1 = 1¢$.

Jean's Proof

$\$1$ can be converted to $100¢$.
100 can be expressed as $(10)^2$.
10 cents is one-tenth of a dollar.
$(0.1)^2 = 0.01$
One hundredth of a dollar is one cent, so $\$1 = 1¢$.

How can Jean's friend Grant show the error in her reasoning?

Grant's Solution

$\$1$ can be converted to $100¢$. ✔

It is true that 100 cents is the same as $1.

100 can be expressed as $(10)^2$. ✔

It is true that $(10)^2$ is $10 \cdot 10$, which is 100.

10 cents is one-tenth of a dollar. ✔

It is true that 10 dimes make up a dollar.

$$(0.1)^2 = 0.01$$ ✔

Arithmetically, I could see that this step was true. But Jean was ignoring the units. It doesn't make sense to square a dime. The units $¢^2$ and $\$^2$ have no meaning.

A dollar is equivalent to $(10)(\$0.10)$ or $10(10¢)$, not to $(10¢)(10¢)$ or $(\$0.10)(\$0.10)$.

$$\$1 \neq 1¢$$

Your Turn

Does Grant's explanation fully show the error in Jean's reasoning? Explain.

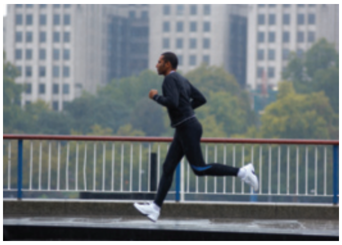
In Summary

Key Idea

- A single error in reasoning will break down the logical argument of a deductive proof. This will result in an invalid conclusion, or a conclusion that is not supported by the proof.

Need to Know

- Division by zero always creates an error in a proof, leading to an invalid conclusion.
- Circular reasoning must be avoided. Be careful not to assume a result that follows from what you are trying to prove.
- The reason you are writing a proof is so that others can read and understand it. After you write a proof, have someone else who has not seen your proof read it. If this person gets confused, your proof may need to be clarified.



Calories burned while running depend on mass, distance, and time. A runner of mass 70 kg who runs 15 km in 1 h will burn about 1000 Calories.

CHECK Your Understanding

1. Determine the error in each example of deductive reasoning.

a) All runners train on a daily basis. Gabriel is a runner. Therefore, Gabriel trains daily.

b) All squares have four right angles. Quadrilateral *PQRS* has four right angles. Therefore, *PQRS* is a square.

2. According to this proof, $5 = 7$. Identify the error.

Proof

$$\begin{aligned} 1 &= 1 + 1 \\ 2(1) &= 2(1 + 1) \\ 2(1) + 3 &= 2(1 + 1) + 3 \\ 2 + 3 &= 4 + 3 \\ 5 &= 7 \end{aligned}$$

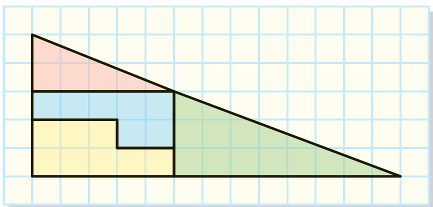
PRACTISING

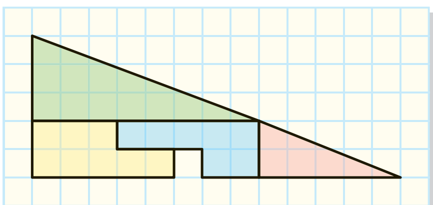
3. Mickey says he can prove that $2 = 0$. Here is his proof.
Let both a and b be equal to 1.

$a = b$	Transitive property
$a^2 = b^2$	Squaring both sides
$a^2 - b^2 = 0$	Subtracting b^2 from both sides
$(a - b)(a + b) = 0$	Factoring a difference of squares
$\dfrac{(a - b)(a + b)}{(a - b)} = \dfrac{0}{(a - b)}$	Dividing both sides by $a - b$
$1(a + b) = 0$	Simplifying
$a + b = 0$	
$1 + 1 = 0$	Substitution
$2 = 0$	

Explain whether each statement in Mickey's proof is valid.

4. Noreen claims she has proved that $32.5 = 31.5$.





Is Noreen's proof valid? Explain.

5. Ali created a math trick in which she always ended with 4. When Ali tried to prove her trick, however, it did not work.

Ali's Proof

n	I used n to represent any number.
$2n$	Multiply by 2.
$2n + 8$	Add 8.
$2n + 4$	Divide by 2.
$n + 4$	Subtract your starting number.

Identify the error in Ali's proof, and explain why her reasoning is incorrect.

6. Connie tried this number trick:

- Write down the number of your street address.
- Multiply by 2.
- Add the number of days in a week.
- Multiply by 50.
- Add your age.
- Subtract the number of days in a year.
- Add 15.

Connie's result was a number in which the tens and ones digits were her age and the rest of the digits were the number from her street address. She tried to prove why this works, but her final expression did not make sense.

Let n represent any house number.

$2n$	Multiply by 2.
$2n + 7$	Add the number of days in a week.
$100n + 350$	Multiply by 50.

Let a represent any age.

$100n + 350 + a$	Add your age.
$100n + 350 + a - 360$	Subtract the number of days in a year.
$100n + a + 5$	Add 15.

a) Try this number trick to see if you get the same result as Connie.
b) Determine the errors in her proof, and then correct them.
c) Explain why your final algebraic expression describes the result of this number trick.

7. According to this proof, $2 = 1$. Determine the error in reasoning.
Let $a = b$.

$a^2 = ab$	Multiply by a.
$a^2 + a^2 = a^2 + ab$	Add a^2.
$2a^2 = a^2 + ab$	Simplify.
$2a^2 - 2ab = a^2 + ab - 2ab$	Subtract $2ab$.
$2a^2 - 2ab = a^2 - ab$	Simplify.
$2(a^2 - ab) = 1(a^2 - ab)$	Factor.
$2 = 1$	Divide by $(a^2 - ab)$.

Closing

8. Discuss with a partner how false proofs can appear to be both reasonable and unreasonable at the same time. Summarize your discussion.

Extending

9. Brittney said she could prove that a strip of paper has only one side. She took a strip of paper, twisted it once, and taped the ends together. Then she handed her friend Amber a pencil, and asked Amber to start at any point and draw a line along the centre of the paper without lifting the pencil. Does a strip of paper have only one side? Why or why not?

10. Brenda was asked to solve this problem:

    Three people enjoyed a meal at a Thai restaurant. The waiter brought a bill for \$30. Each person at the table paid \$10.

    Later the manager realized that the bill should have been for only \$25, so she sent the waiter back to the table with \$5.

    The waiter could not figure out how to divide \$5 three ways, so he gave each person \$1 and kept \$2 for himself.

    Each of the three people paid \$9 for the meal.

    $9 \cdot 3 = 27$

    The waiter kept \$2.

    $27 + 2 = 29$

    What happened to the other dollar?

Does the question make sense? How should Brenda answer it?

1.6 Reasoning to Solve Problems

GOAL

Solve problems using inductive or deductive reasoning.

YOU WILL NEED

- calculator

INVESTIGATE the Math

Emma was given this math trick:

- Choose a number.
- Multiply by 6.
- Add 4.
- Divide by 2.
- Subtract 2.

Emma was asked to use inductive reasoning to make a conjecture about the relationship between the starting and ending numbers, and then use deductive reasoning to prove that her conjecture is always true. Here is her response to the problem:

Inductive reasoning:

#	×6	+4	÷2	−2
5	30	34	17	15
−3	−18	−14	−7	−9
0	0	4	2	0
24	144	148	74	72

I followed the steps to work through four examples.

Conjecture: It is 3 times.

Deductive reasoning:

I chose *d*.

Then I multiplied, added, divided, and subtracted to get an expression.

$$\left(\frac{6d + 4}{2}\right) - 2$$

It worked.

It simplified to $3d$.

EXPLORE...

- Suppose that you are lost in the woods for hours and come upon a cabin. In the cabin, you find a lantern, a candle, a wood stove with wood in it, and a match. What do you light first?

? How can Emma's communication about her reasoning be improved?

A. With a partner, explain why Emma might have chosen the values she did.

B. What details are missing from the deductive reasoning Emma used to arrive at the expression $3d$?

C. Improve Emma's conjecture, justifications, and explanations.

Reflecting

D. How does it help to understand the mathematics when both symbols and words are used in an explanation?

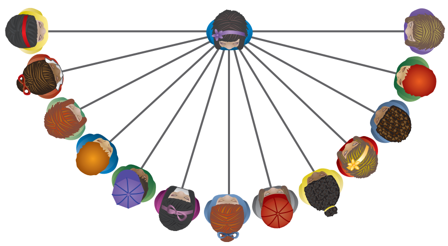
E. Why is it important to explain your reasoning clearly?

APPLY the Math

EXAMPLE 1 Using reasoning to solve a problem

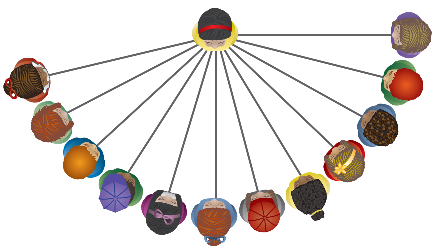
The members of a recently selected varsity basketball team met each other at their first team meeting. Each person shook the hand of every other person. The team had 12 players and 2 coaches. How many handshakes were exchanged?

Kim's Solution



I decided to think about how many times each person shook hands. There were 14 people in total, so person 1 shook hands with each of the other 13 people.

13 handshakes



Person 2 had already shaken hands with person 1. Person 2 shook hands with each of the remaining 12 people.

$13 + 12$ handshakes

$13 + 12 + 11 + 10 + 9 + 8 + 7$
$+ 6 + 5 + 4 + 3 + 2 + 1$
$= 91$ handshakes

This pattern of handshakes continued until there were two people left when the last handshake happened.

Your Turn

Discuss, with a partner, whether Kim used inductive or deductive thinking in her solution. How do you know?

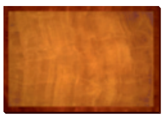
EXAMPLE 2 Using reasoning to solve a problem

Sue signed up for games at her school's fun night. Seven other people were assigned to her group, making up four pairs of partners. The other members of her group were Dave, Angie, Josh, Tanya, Joy, Stu, and Linus. When the games started, Dave and his partner were to the left of Stu. Across from Dave was Sue, who was to the right of Josh. Dave's brother's partner, Tanya, was across from Stu. Joy was not on Stu's right.

Name the four pairs of partners.

Vicky's Solution

Dave
Angie
Josh
Tanya
Joy
Stu
Linus
Sue



I drew a rectangle to represent a table. I made a list of the students' names so I could cross them off as I put them in place.

~~Dave~~
Angie
Josh
Tanya
Joy
~~Stu~~
Linus
Sue

The first names I wrote in were Dave and Stu, since they were the first two mentioned. It didn't matter where I started, as long as I kept the relationships of left, right, and across the table. I crossed Dave and Stu off my list.

~~Dave~~
Angie
~~Josh~~
Tanya
Joy
~~Stu~~
Linus
~~Sue~~

I knew that Sue was across from Dave and to the right of Josh. I crossed Sue and Josh off my list.

~~Dave~~
Angie
~~Josh~~
~~Tanya~~
Joy
~~Stu~~
~~Linus~~
~~Sue~~

The next clue mentioned that Dave's brother and his partner Tanya were across from Stu. The only male name left was Linus, so Linus and Tanya were partners. I crossed their names off my list.

If Joy was not on Stu's right, then she must have been on his left. Therefore, she must have been Dave's partner. So, the last person to match was Angie with Sue.

The four pairs of partners were Linus and Tanya, Dave and Joy, Sue and Angie, and Stu and Josh.

The partners sat together, on the same side of the table.

Your Turn

Discuss with a partner whether inductive or deductive reasoning was used for this solution. How do you know?

In Summary

Key Idea

- Inductive and deductive reasoning are useful in problem solving.

Need to Know

- Inductive reasoning involves solving a simpler problem, observing patterns, and drawing a logical conclusion from your observations to solve the original problem.
- Deductive reasoning involves using known facts or assumptions to develop an argument, which is then used to draw a logical conclusion and solve the problem.

CHECK Your Understanding

1. Explain which type of reasoning is demonstrated by each statement.

a) Over the past 12 years, a tree has produced plums every other year. Last year, the tree did not produce plums. Therefore, the tree will produce plums this year.

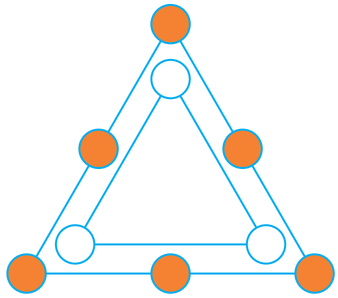
b) Mammals have hair. Dogs are mammals. Therefore, dogs have hair.

c) Every Thursday, a train arrives at 2:30 p.m. Today is Thursday, so the train will arrive at 2:30 p.m.

d) Every even number has a factor of 2. 24 is an even number. Therefore, 24 has a factor of 2.

e) For the pattern 3, 12, 21, 30, 39, the next term is 48.

2. Copy this diagram. Place the digits 1 through 9 in the circles so that the sum of the numbers on the outside triangle is double the sum of the numbers on the inside triangle. Explain whether more than one solution is possible.



PRACTISING

3. Draw the next figure in this sequence.

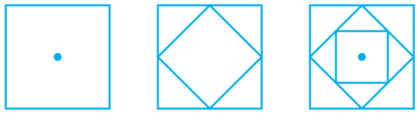


Figure 1 Figure 2 Figure 3

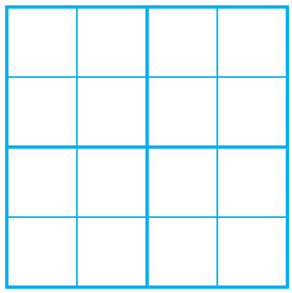
4. a) Substitute numbers for the letters to create an addition problem with a correct answer.

b) How many solutions are possible?

$$\begin{array}{r} y \\ xxx \\ xxx \\ xxx \\ +\ xxx \\ \hline yxxx \end{array}$$

5. a) Choose four different colours. Fill in the cells in a copy of this chart, so that each row and column has four different colours and each quadrant also has four different colours.

b) Compare your strategy with a classmate's strategy. How are your strategies the same? How are they different?



6. A farmer wants to get a goat, a wolf, and a bale of hay to the other side of a river. His boat is not very big, so it can only carry him and one other thing. If the farmer leaves the goat alone with the bale of hay, the goat will eat the hay. If he leaves the wolf alone with the goat, the wolf will eat the goat. When the farmer is present, the goat and the hay are safe from being eaten. How does the farmer manage to get everything safely to the other side of the river?

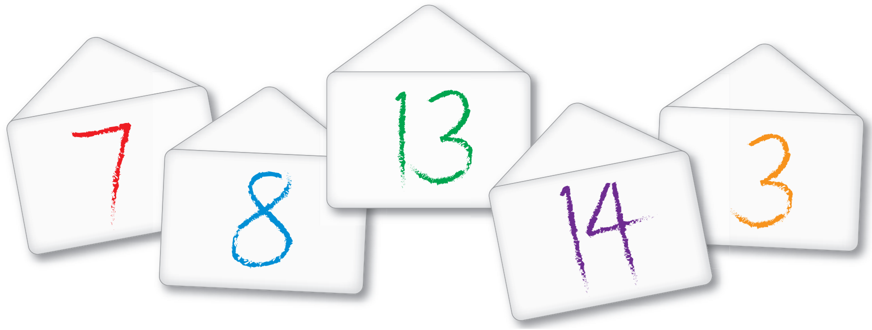
Goats have the reputation that they will eat almost anything. In fact, they will taste just about anything, but they are picky about what they eat. They do eat hay.

7. Determine the unknown term in this pattern: 17, 22, ___, 35, 43. Explain your reasoning.

8. Suppose that you are marooned on an island where there are only liars and truth-tellers. Liars always tell lies, and truth-tellers always tell the truth. You meet two siblings. The brother says, "My sister told me that she is a liar." Is he a liar or a truth-teller? Explain how you know.

9. Bob, Kurt, and Morty are football players. One is a quarterback, one is a receiver, and one is a kicker. The kicker, who is the shortest of the three, is not married. Bob, who is Kurt's father-in-law, is taller than the receiver. Who plays which position?

10. A set of 10 cards, each showing one of the digits from 0 to 9, is divided between five envelopes so that there are two cards in each envelope. The sum of the cards inside each envelope is written on the envelope:



A sum of 8 could be made by these pairs of cards: (8, 0), (7, 1), (6, 2), and (5, 3).

a) Explain which of these pairs of cards cannot possibly be in the envelope marked 8.

b) Describe the reasoning you used to solve this problem.

11. Solve the multiplication problem below. Each letter represents a different digit, and the product is correct.

$abcd \cdot 4 = dcba$

12. At lunchtime, a soccer team meets in the school cafeteria to help organize a tournament. There are 18 players and 2 coaches at the meeting. The tables in the cafeteria are rectangular. Two people can sit on each of the long sides, and one person can sit at each end.

a) What arrangement of tables would enable the team members to sit as close to each other as possible, so that everyone can be heard?

b) Compare your solution with other students' solutions. As a group, decide which is the best solution for the team.

Competitors in the Eco-Challenge race 500 km through the mountains of British Columbia.

13. Early in a bicycle race, Tamara led Kateri by 3 km, while Justine was behind Shreya by 2 km. Shreya was ahead of Kateri by 1 km. By the halfway point, Tamara and Shreya had exchanged places, but they were still the same distance apart. Justine had pulled even with Tamara. Over the last part of the race, Justine dropped 1 km behind Tamara, and Kateri passed Shreya; there were no other changes of position. Who finished third?

14. Use inductive reasoning to determine the number of diagonals that can be drawn in a decagon (a polygon with 10 sides).

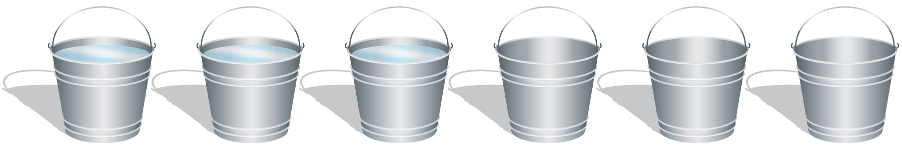
15. Max, Karl, Terri, and Suganthy live on the first floor of an apartment building. One is a manager, one is a computer programmer, one is a singer, and one is a teacher.

a) Use the statements below to determine which person is the manager.

- Suganthy and Terri eat lunch with the singer.
- Karl and Max carpool with the manager.
- Terri watches football on television with the manager and the singer.

b) Describe the reasoning you used to solve this problem.

16. There are six pails in a row. The first three pails are filled with water. How can you move only one pail to make the following pattern: full pail, empty pail, full pail, empty pail, full pail, empty pail?



Closing

17. How do you recognize a problem that can be solved using inductive reasoning? How do you recognize a problem that can be solved using deductive reasoning? Is it always possible to tell which kind of reasoning is needed to solve a problem? Explain.

Extending

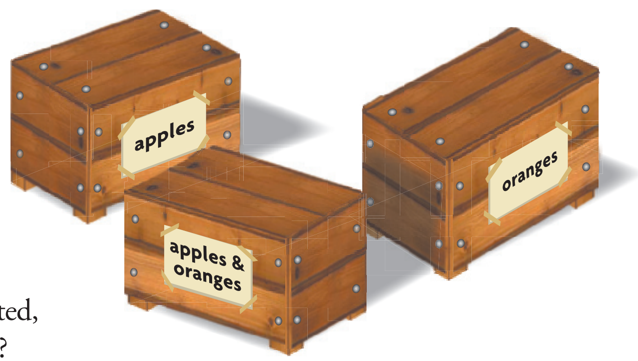
18. During Sid's vacation, it rained on five days. However, when it rained in the morning, the afternoon was sunny, and every rainy afternoon was preceded by a sunny morning. There were six sunny mornings and nine sunny afternoons. How long was Sid's vacation?

19. Two girls, Arlene and Cathy, and two boys, Leander and Dean, are athletes. One is a long distance runner, one is a softball player, one is a hockey player, and one is a golfer. At lunchtime, they sit around a square table, usually in the same places.

- The runner sits on Arlene's left.
- The hockey player sits across from Leander.
- Cathy and Dean sit next to each other.
- A girl sits on the softball player's left.

Who is the golfer?

20. The labels have been placed on the wrong boxes. You may select one fruit from one box, but you may not look in the box. Based on the fruit you have selected, how can you immediately label all the boxes correctly?



1.7 Analyzing Puzzles and Games

YOU WILL NEED
- counters or coins

GOAL

Determine, explain, and verify a reasoning strategy to solve a puzzle or to win a game.

EXPLORE...
- Three students are playing a game. Two of the students flip a coin, and the third student records their scores. Student 1 gets a point if the result is two heads, student 2 gets a point if the result is two tails, and student 3 gets a point if the result is a head and a tail. The first student to get 10 points wins. Explain whether you would prefer to be student 1, student 2, or student 3.

INVESTIGATE the Math

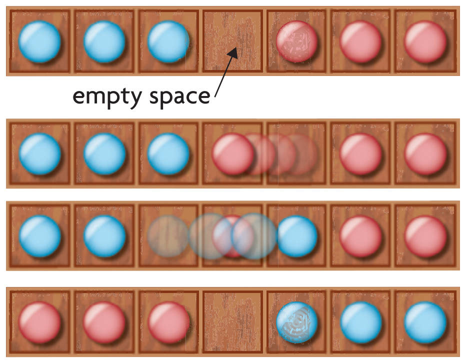
To solve a leapfrog puzzle, coloured counters are moved along a space on a board. The goal is to move each set of coloured counters to the opposite side of the board.

Board at start

A counter can move into the empty space.

A counter can leapfrog over another counter into the empty space.

Board at end



? What is the minimum number of moves needed to switch five counters of each colour?

A. Develop a group strategy to switch the blue and red counters using as few moves as possible.

B. Execute your strategy, counting each move you make.

C. How many moves did you need to complete the switch?

Reflecting

D. How did you know that you had completed the switch in the fewest number of moves?

E. Did you use inductive or deductive reasoning to solve the puzzle? Explain.

F. Predict the minimum number of moves needed to solve the puzzle if you had six counters of each colour. Explain how you made your prediction.

G. Did you use inductive or deductive reasoning in step F? Explain.

APPLY the Math

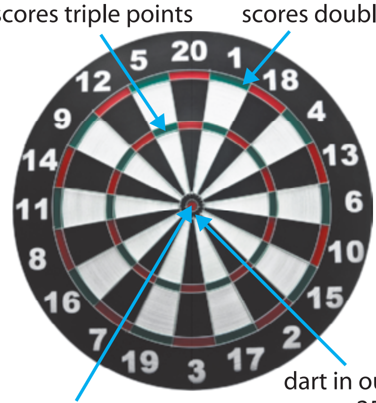
EXAMPLE 1 Using reasoning to determine possible winning plays

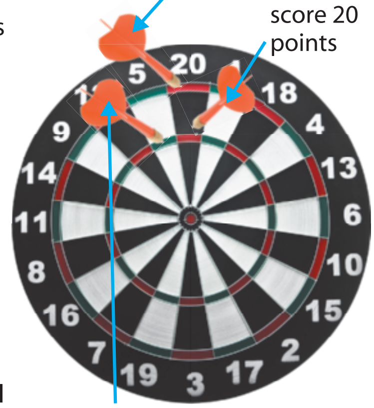
Frank and Tara are playing darts, using the given rules. Their scores are shown in the table below. To win, Frank must reduce his score to exactly zero and have his last counting dart be a double.

Rules

- Each player's score starts at 501.
- The goal is to reduce your score to zero.
- Players alternate turns.
- Each player throws three darts per turn.

Frank		Tara	
Turn Score	Total Score	Turn Score	Total Score
	501		501
100	401	85	416
95	306	85	331
140	166	140	191
130	36	91	100





Total score for turn: 75 points

What strategies for plays would give Frank a winning turn?

Frank's Solution

$2(18) = 36$ — I could win with a single dart in double 18.

$18 + 2(9) = 36$ — If I hit 18 instead of double 18, then I could use my second dart to try for double 9.

$18 + 9 = 27$
9 would be left. — If I hit 9 instead of double 9 with my second dart, then I couldn't win this turn. That's because I can't score 9 with a double.

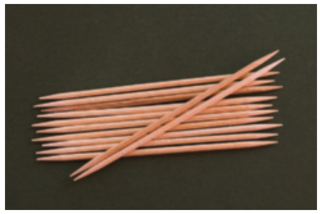
Your Turn

a) Describe two other ways that Frank could win the game on his turn.

b) If Frank does not win on his turn, describe a strategy that Tara could use to win on her next turn.

EXAMPLE 2 Using deductive and inductive reasoning to determine a winning strategy

Nadine and Alice are playing a toothpick game. They place a pile of 20 toothpicks on a desk and alternate turns. On each turn, the player can take one or two toothpicks from the pile. The player to remove the last toothpick is the winner. Nadine and Alice flip a coin to determine the starting player.



Is there a strategy Alice can use to ensure that she wins the game?

Alice's Solution

I need to make sure that there are one or two toothpicks left after Nadine's last turn.

I will win the game if I can take the last toothpick. If I work backward, I might see a pattern I can use to win.

To make sure this happens, I have to leave three toothpicks on the desk for Nadine.

If I leave three toothpicks, Nadine has to take either one or two toothpicks. If she takes only one, I can take the two that are left and win. If she takes two, I can take the last one and win.

To make sure this happens, I have to leave six toothpicks on the desk for Nadine.

If I leave six toothpicks, Nadine has to take either one or two toothpicks. If she takes only one, I can take two, which would leave three. If she takes two, I can take one and leave her with three.

To make sure this happens, I have to leave nine toothpicks on the desk for Nadine.

If I leave nine toothpicks, Nadine has to take either one or two toothpicks. If she takes only one, then I can take two, which would leave six. If she takes two, I can take one and leave her with six.

I can see that I need to leave 12, 15, and 18 toothpicks for Nadine.

There is a pattern to the number of toothpicks I must leave for Nadine: 3, 6, 9, 12, 15, 18.

I will win if I go first and take two toothpicks. Each turn after that, I need to pick one or two so that I leave Nadine with a number of toothpicks that is a multiple of 3.

If Nadine goes first and knows this strategy, I can't win. If she goes first and doesn't know this strategy, however, I can win by arranging to leave her a number of toothpicks that is a multiple of 3.

Your Turn

a) Which part of Alice's strategy involved deductive reasoning? Explain.

b) Which part of Alice's strategy involved inductive reasoning? Explain.

In Summary

Key Idea

- Both inductive reasoning and deductive reasoning are useful for determining a strategy to solve a puzzle or win a game.

Need to Know

- Inductive reasoning is useful when analyzing games and puzzles that require recognizing patterns or creating a particular order.
- Deductive reasoning is useful when analyzing games and puzzles that require inquiry and discovery to complete.

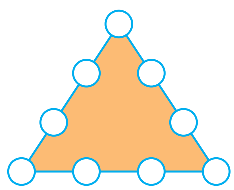
CHECK Your Understanding

1. In the leapfrog puzzle, what would be the minimum number of moves needed to exchange 10 red counters with 10 blue counters? Explain how you know.
2. Frank and Tara are playing another game of darts. Tara's game score is 66. List three different strategies she could use to win on her turn.
3. In the toothpick game, suppose that players are allowed to take one, two, or three toothpicks. Determine a strategy you could use to ensure that you win if you do not have the first turn.

PRACTISING

4. Rearrange three golf balls so that the arrowhead points down instead of up.

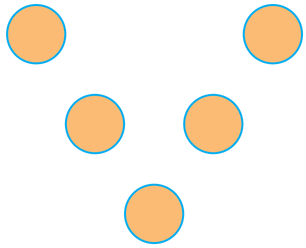
5. **a)** Draw a diagram like the one to the right. Place the numbers 1 through 9 in the circles so that the sum of the numbers on each side of the triangle is 17.
   b) Explain the strategy you used.



6. Examine this square. It has a magic sum.
   a) Describe the patterns you see.
   b) Selva noticed that when he added three numbers that were not in the same row or column, the sum was 36 (the magic sum). This number is 3 times the number in the middle square. Create your own magic square using the patterns you identified. Do Selva's observations hold in your square?
   c) Prove algebraically that Selva's observations hold in any square that is created using these patterns.

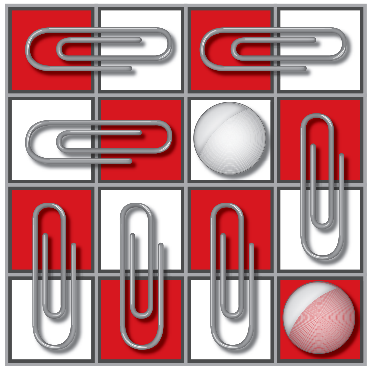
5	9	13
8	12	16
11	15	19

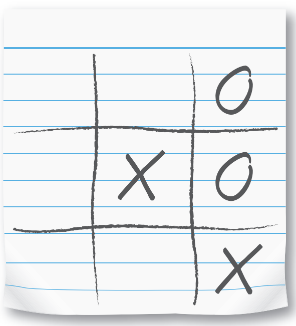
7. Place the numbers 1 to 5 in a V shape, as shown, so that the two arms of the V have the same total.
   a) How many different solutions are there?
   b) What do you notice about all the solutions you found?
   c) How could you convince someone that you have identified all the possible solutions?



8. Draw a 4-by-4 grid that is large enough to place a coin in any square. Your opponent in this game has seven paper clips. Each paper clip is large enough to cover two squares when placed horizontally or vertically. You need to place a coin on each of any two squares so that your opponent is unable to cover the remaining squares with the seven paper clips. Determine a strategy to ensure that you will always win.

The player with the paper clips wins.





9. Who started this game of tic-tac-toe: player X or player O? Explain. Assume that both players are experienced at playing tic-tac-toe.

10. Sudoku requires both inductive and deductive reasoning skills. The numbers that are used to complete a Sudoku puzzle relate to the size of the grid. For a 6-by-6 grid, the numbers 1 to 6 are used. For a 9-by-9 grid, the numbers 1 to 9 are used. The grid must be filled so that each column, row, or block contains all the numbers. No number can be repeated within any column, row, or block. Solve each of the Sudoku puzzles below.

a)

5			2	6	
		4			
1					6
			1	5	
	3			2	1
			6		

b)

6		4	8	2				
					1	4		
				6		3	5	
				1			4	
8	9	2					1	
	1	3	2	9				
4			5					6
5						9	3	

11. Fill in the missing numbers, from 1 to 9, so that the sum of the numbers in each row, column, and diagonal is 15.

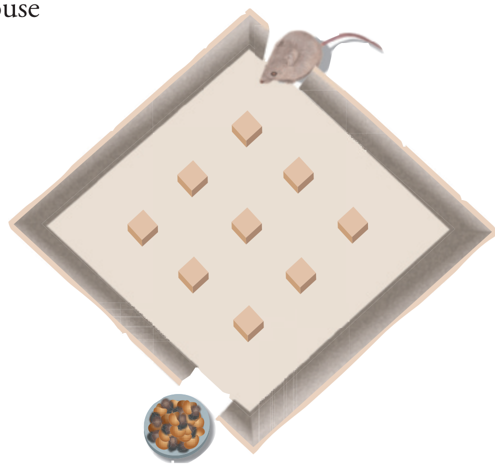
a)

		6
		1
4	3	8

b)

		4
	5	3
		8

12. How many ways can the mouse navigate the maze to reach the trail mix, if the mouse can only travel down?



13. KenKen, like Sudoku, requires both inductive and deductive reasoning skills. Solve this 6×6 KenKen puzzle using only the numbers 1 to 6. Do not repeat a number in any row or column. The darkly outlined sets of squares are cages. The numbers in each cage must combine in any order to produce the target number, using the operation shown. For example, the target in the top left cage is $30 \times$, which means 30 by multiplication. The two numbers in the cage must be 5 and 6, because no other combination of two factors (from 1 to 6) gives a product of 30. A number may be repeated in a cage as long as it is not in the same row or column.

$30 \times$		$36 \times$	$2 \div$		$18 +$
$3 +$			$7 +$		
	$20 \times$		$5 -$		
$1 -$	$2 -$			$13 +$	
	$7 +$		$2 -$		
$2 \div$			$3 -$		

Closing

14. Explain how inductive and deductive reasoning can help you develop a strategy to play a game or to solve a puzzle.

Extending

15. a) Suppose that the goal for tic-tac-toe is changed, so that you have to force your opponent to place three markers in a row, column, or diagonal in order to win. How would your strategy change?

b) What role does inductive and deductive reasoning play in helping you develop your new strategy?

1 Chapter Self-Test

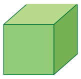


Figure 1

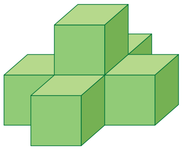


Figure 2

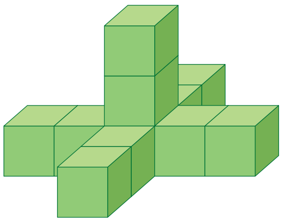


Figure 3

1. Danielle made the cube structures to the left.

a) What would the 4th and 5th structures look like? How many cubes would Danielle need to build each of these structures?

b) Make a conjecture about the relationship between the nth structure and the number of cubes needed to build it.

c) How many cubes would be needed to build the 25th structure? Explain how you know.

2. Frank tosses a coin five times, and each time it comes up tails. He makes the following conjecture: The coin will come up tails on every toss. Is his conjecture reasonable? Explain.

3. Koby claims that the perimeter of a pentagon with natural number dimensions will always be an odd number. Search for a counterexample to his claim.

4. Prove that the product of two consecutive odd integers is always odd.

5. Prove that the following number trick always results in 10:
Choose a natural number. Double it. Add 20. Divide by 2. Subtract the original number.

6. Andy, Bonnie, Candice, and Darlene are standing in line to buy ice cream. Determine the order in which they are lined up, using these clues:
- Candice is between Andy and Bonnie.
- Darlene is next to Andy.
- Bonnie is not first.

7. The following proof seems to show that $10 = 9.\overline{9}$. Is this proof valid? Explain.
Let $a = 9.\overline{9}$.

$10a = 99.\overline{9}$	Multiply by 10.
$10a - a = 90$	Subtract a.
$9a = 90$	Simplify.
$a = 10$	Divide by 9.

WHAT DO You Think Now? Revisit **What Do You Think?** on page 5. How have your answers and explanations changed?

Chapter Review

FREQUENTLY ASKED Questions

Q: Why is it important to check carefully any proof you develop and the associated reasoning you used?

A: If you have made a statement using flawed reasoning, you can end up with a conclusion that does not make sense. For example:
All pitbulls are ferocious dogs.
Jake is a pitbull.
Jake is a ferocious dog.

In this example, the premise is faulty. Pitbulls are known to be a ferocious breed, but we cannot be certain that every pitbull in the world is a ferocious dog. This leads to the invalid conclusion that Jake is ferocious.

Study *Aid*

- See Lesson 1.5, Examples 1 and 2.
- Try Chapter Review Questions 11 and 12.

Q: What can deductive reasoning and inductive reasoning be used for?

A1: Inductive reasoning can help you make conjectures. You can look for patterns in several examples or cases. The patterns you observe in specific cases can be generalized to a statement that includes all cases. This forms the basis for a conjecture. Deductive reasoning gives you the ability to prove that your conjecture is valid for every possible case.

For example, when you examine the sum of three consecutive natural numbers, you can see a pattern.
$1 + 2 + 3 = 6$
$2 + 3 + 4 = 9$
$3 + 4 + 5 = 12$
$4 + 5 + 6 = 15$

Each sum is a multiple of 3. Based on this pattern, you can make the following conjecture: The sum of three consecutive natural numbers is divisible by 3. You can then use deductive reasoning to prove your conjecture.

$x, x + 1, x + 2$	Let x, $x + 1$, and $x + 2$ represent three consecutive natural numbers.
$x + x + 1 + x + 2$	Write an expression for the sum.
$3x + 3$	Collect like terms.
$3(x + 1)$	Factor.
$3(x + 1)$ is divisible by 3.	Since 3 is a factor of the expression, the expression will always be divisible by 3.

Study *Aid*

- See Lesson 1.6, Examples 1 and 2, and Lesson 1.7, Examples 1 and 2.
- Try Chapter Review, Questions 12 to 15.

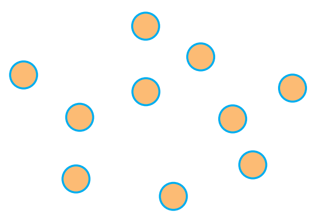
A2: Inductive and deductive reasoning are useful strategies for solving problems, some types of puzzles, and some types of games.

Word puzzle: Identify a five-letter word correctly in five or fewer attempts. After each attempt, the person who knows the word will show you the correct letters in the correct positions in yellow, the correct letters in the wrong positions in red, and the incorrect letters with no colour.

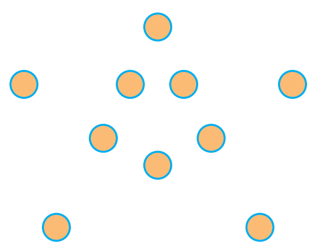
	Response
My first guess is RENTS, since it has letters that are often used in English words.	RENTS
I know that S is in the correct position. R is in the word but in a different position. E, N, and T are not in the word. The letter R is often used after the letters C, D, F, G, and P, so that could put R in the second position. I chose DRAWS as my next guess, because A is the next most common vowel.	DRAWS
I know that the vowel has to be I, O, or U. I also know that the consonants D, R, and W are in the word, but in the wrong positions in DRAWS. From my guesses, I think that R goes in the third position since I can't think of a word that would have either _WDRS or _DWRS. Also, from the patterns of letters, I think the R and D go together. That would make the unknown word to be W_RDS. The only vowel that would fit is O. My guess is WORDS.	WORDS

Problem: Make five lines of counters, with four counters in each line, using only 10 counters.

Since an array of (5)(4) counters would require 20 counters, I deduce that most counters must be part of more than one line. This means that the lines must overlap.



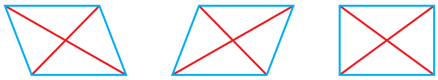
I can also deduce that the lines need to overlap more than once since there are only 10 counters available. Lines of four counters could overlap at any counter.



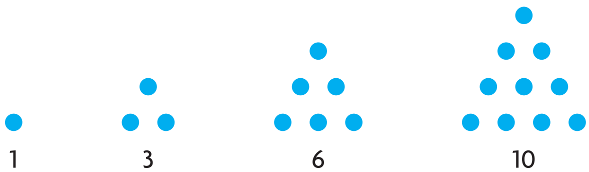
PRACTISING

Lesson 1.1

1. Charles studied the diagonals of the parallelograms below to look for patterns. Make a conjecture about the diagonals of parallelograms. What evidence supports your conjecture?



2. Consider the following sequence of triangular numbers:



a) Describe the pattern, and use it to determine the next four triangular numbers.

b) Consider the products $1 \cdot 2$, $2 \cdot 3$, $3 \cdot 4$, $4 \cdot 5$. Explain how these products are related to each triangular number.

c) Make a conjecture about a formula you could use to determine the *n*th triangular number.

3. Examine the following number pattern:

$1^3 = 1$ and $1 = 1^2$

$1^3 + 2^3 = 9$ and $9 = 3^2$

$1^3 + 2^3 + 3^3 = 36$ and $36 = 6^2$

$1^3 + 2^3 + 3^3 + 4^3 = 100$ and $100 = 10^2$

a) Describe the pattern you see.

b) Use your observations to predict the next equation in the pattern.

c) Make a conjecture about the sum of the first n cubes.

Lesson 1.2

4. a) Examine this pattern to determine the next equation:

$37 \times 3 = 111$

$37 \times 6 = 222$

$37 \times 9 = 333$

$37 \times 12 = 444$

b) Is your conjecture correct? Explain how you know.

c) This pattern eventually breaks down. Determine when the breakdown occurs.

Lesson 1.3

5. a) What is a counterexample?

b) Explain why counterexamples are useful.

6. Harry claims that if opposite sides of a quadrilateral are the same length, the quadrilateral is a rectangle. Do you agree or disagree? Justify your decision.

7. Sadie claims that the difference between any two positive integers is always a positive integer. Do you agree or disagree? Justify your decision.

Lesson 1.4

8. Complete the conclusion for the following deductive argument: If an integer is an even number, then its square is also even. Six is an even number, therefore, ...

9. Prove that the product of two odd integers is always odd.

10. Linda came across this number trick on the Internet and tried it:

- Think about the date of your birthday.
- Multiply the number for the month of your birthday by 5. (For example, the number for November is 11.)
- Add 7.
- Multiply by 4.
- Add 13.
- Multiply by 5.
- Add the day of your birthday.
- Subtract 205.
- Write your answer.

a) Try the trick. What did you discover?

b) Prove how this trick works. Let m represent the number for the month of your birthday and d represent the day.

11. Examine the relationships below.

$2(3^2 + 5^2) = 2^2 + 8^2$
$2(2^2 + 3^2) = 1^2 + 5^2$
$2(7^2 + 4^2) = 3^2 + 11^2$

a) Describe the patterns you see.

b) Jen makes the following conjecture: If you double the sum of two squares, the product is always the sum of two squares. Prove Jen's conjecture.

Lesson 1.5

12. The following proof seems to show that $2 = 1$. Examine this proof, and determine where the error in reasoning occurred.

Let $a = b$.

$a^2 = ab$	Multiply by a.
$a^2 - b^2 = ab - b^2$	Subtract b^2.
$(a - b)(a + b) = b(a - b)$	Factor.
$a + b = b$	Divide by $(a - b)$.
$b + b = b$	$a = b$
$2b = b$	Simplify.
$2 = 1$	Divide by b.

13. Julie was trying to prove that a number trick always results in 5:

Julie's Proof

n	Choose a number.
$n + 10$	Add 10.
$5n + 10$	Multiply the total by 5.
$5n - 40$	Subtract 50.
$\frac{5n - 40}{n}$	Divide by the number you started with.

Identify the error in Julie's proof, and correct it.

Lesson 1.6

14. Two mothers and two daughters got off a city bus, reducing the number of passengers by three. Explain how this is possible.

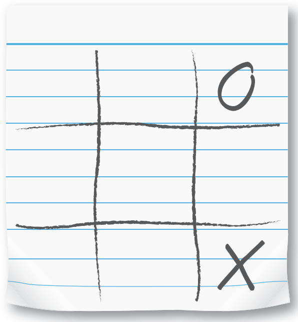
15. The three little pigs built three houses: one of straw, one of sticks, and one of bricks. By reading the six clues, deduce which pig built each house, the size of each house, and the town in which each house was located.

Clues

- Penny Pig did not build a brick house.
- The straw house was not medium in size.
- Peter Pig's house was made of sticks, and it was neither medium nor small in size.
- Patricia Pig built her house in Pleasantville.
- The house in Hillsdale was large.
- One house was in a town called Riverview.

Lesson 1.7

16. If you are playing next in this game of tic-tac-toe, you can use a strategy that guarantees you will win the game. Explain what this strategy is.



17. The rules for the game of 15 are given below:

- The cards are placed on a table between two players.
- Players take turns choosing a card (any card they like).
- The winner is the first player to have three cards that add to 15.
  For example, if you drew 1, 5, 6 and 8, then you would win, because $1 + 6 + 8 = 15$.

a) Is it possible to win the game in three moves?

b) Devise a winning strategy. Explain your strategy.

Chapter Task

How Many Sisters and Brothers?

Rob, Yu, and Wynn challenged each other to create a number trick that ended with the number of siblings they have. Their number tricks are given below.

Rob's Number Trick

- Choose a number.
- Add 3.
- Multiply by 2.
- Subtract 2.
- Multiply by 5.
- Divide by 10.
- Add 3.
- Subtract the starting number.

Wynn's Number Trick

- Choose a number.
- Multiply by 4.
- Add 8.
- Divide by 4.
- Subtract the starting number.

Yu's Number Trick

- Choose a number.
- Subtract 2.
- Multiply by 0.
- Divide by 5.

? What would your number trick be?

A. Create a number trick that always ends with the number of siblings you have. Use at least three different operations and at least four steps. Test your number trick to make sure that it works.

B. Trade number tricks with a classmate, and test your classmate's trick at least three times.

C. Make a conjecture about the number of siblings your classmate has.

D. Use deductive reasoning to develop a proof of your conjecture.

E. Is there a number that will not work in your number trick? Explain.

Task | ***Checklist***

✔ Are the steps in your number trick clear?

✔ Did your conjecture correctly predict the result of your classmate's trick?

✔ Are the statements in your proof valid and clear?

✔ Did you provide your reasoning?

1 Project Connection

Creating an Action Plan

Deadlines are part of life. Completing projects on time is just as important in the workplace as it is in school. So, how can you avoid having to rush through all the stages of your research project at the last minute? One way is to use a strategy called backward planning: develop a formal action plan, and create a timeline based on this action plan.

A major research project must successfully pass through several stages. On the next page is an outline for an action plan, with a list of these stages. Completing this action plan will help you organize your time and give you goals and deadlines you can manage. The times that are suggested for each stage are only a guide, with one day equivalent to any regular day in your life. Adjust the time you will spend on each stage to match the scope of your project. For example, a project based on primary data (data that you collect) will usually require more time than a project based on secondary data (data that other people have collected and published). You will also need to consider your personal situation—the issues that are affecting you and may interfere with completion of the project.

Issues Affecting Project Completion

Consider the issues that may interfere with completion of the project in a time-efficient manner. For example:

- part-time job
- after-school sports and activities
- regular homework
- assignments for other courses
- tests in other courses
- driving school
- time you spend with friends
- school dances and parties
- family commitments
- access to research sources and technology

What other issues can you add to this list?

Your Turn

A. Take some time to complete an action plan for your project. Start by deciding on the probable length of time for each stage. Do not forget to include buffer space in your action plan. Buffer space is not a stage, but it is important. If something goes wrong (for example, if you are unable to gather appropriate data for your topic and must select a new topic), having that buffer space in your action plan may allow you to finish your project on time, without making extraordinary efforts.

1. **Select the topic you would like to explore.**
   Suggested time: 1 to 3 days
   Your probable time:
   Finish date:

2. **Create the research question that you would like to answer.**
   Suggested time: 1 to 3 days
   Your probable time:
   Finish date:

3. **Collect the data.**
   Suggested time: 5 to 10 days
   Your probable time:
   Finish date:

   Buffer space
   Suggested time: 3 to 7 days
   Your probable time:
   Finish date:

4. **Analyze the data.**
   Suggested time: 5 to 10 days
   Your probable time:
   Finish date:

5. **Create an outline for your presentation.**
   Suggested time: 2 to 4 days
   Your probable time:
   Finish date:

6. **Prepare a first draft.**
   Suggested time: 3 to 10 days
   Your probable time:
   Finish date:

7. **Revise, edit, and proofread.**
   Suggested time: 3 to 5 days
   Your probable time:
   Finish date:

8. **Prepare and practise your presentation.**
   Suggested time: 3 to 5 days
   Your probable time:
   Finish date:

B. Use a calendar and your probable times for each stage to work backwards from the presentation date to create a schedule you can follow. This will ensure that you will be able to complete all the stages of your project in the time available. In your schedule, include regular conferences with your teacher—5 to 10 min to discuss your progress.