8.4 Scale Factors and Areas of 2-D Shapes

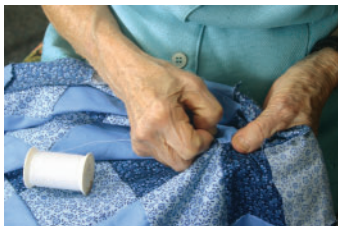
GOAL

Solve area problems that involve similar 2-D shapes.

YOU WILL NEED

- calculator
- ruler
- pattern blocks

INVESTIGATE the Math



Quilting is as old as ancient Egypt, if not older. For most of its history, however, quilting was used to make clothing. Pieced quilts, made by sewing pieces of fabric into blocks and then sewing together the blocks, are a more recent development.

Norma is making a quilt by sewing together congruent pieces of cloth. To create a larger quilt, she sews together more congruent shapes. She makes sure that the larger quilt is similar in shape to the original quilt.

? How does the area of the larger quilt relate to the area of the original quilt?

A. Suppose that Norma uses square pieces of fabric. Use square pattern blocks to represent these pieces of fabric.

Measure the dimensions of one square, and determine its area. Create a table like the one below, and record the dimensions and area.

Length (in.)	Width (in.)	Area (in.^{2})

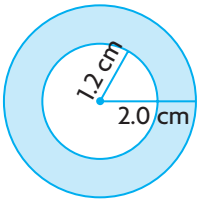
B. Starting with one square, add enough squares to create a larger square that has double the original dimensions. Record its dimensions and area in your table.

C. Add more squares to create a larger square that has three times the original dimensions. Record the dimensions and area of the larger square in your table.

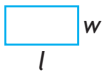
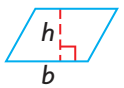
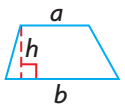
D. Predict the area of a square that has four times the original dimensions. Check your prediction, and record the dimensions and area of this square in your table.

E. As the square grows larger, how does its area relate to the scale factor k and the area of the original square?

EXPLORE...



- Determine the area of the shaded region. If both radii are doubled, does the area also double?

Area Formulas	
Shape	**Formula**
triangle 	$A = \frac{1}{2}bh$
rectangle 	$A = lw$
square 	$A = s^2$
parallelogram 	$A = bh$
trapezoid 	$A = \frac{1}{2}h(a + b)$
circle 	$A = \pi r^2$

F. Suppose that Norma uses congruent triangular pieces of fabric. Repeat parts A to E using triangular pattern blocks and a table like the one below.

Base (in.)	Height (in.)	Area (in.^{2})

G. Suppose that Norma uses congruent rectangular pieces of fabric. Repeat parts A to E using pieces of U.S. letter or U.S. legal paper and a table like the one shown in part A.

H. Make a **conjecture** about the relationship among the area of a shape, the scale factor, and the area of a larger similar shape.

Reflecting

I. Do you think your conjecture will hold when you decrease the dimensions of a shape by a specific scale factor? Explain.

J. Do you think your conjecture will hold for other similar shapes, such as parallelograms, trapezoids, or circles? Explain.

K. Do you think your conjecture will hold for any pair of similar 2-D shapes? Explain.

APPLY the Math

EXAMPLE 1 Reasoning about scale factor and area

Jasmine is making a kite from a 2:25 scale diagram. The area of the scale diagram is 20 cm^2. How much fabric will she need for her kite?

Jasmine's Solution: Reasoning about scale as an enlargement

Scale factor $= \frac{25}{2}$ or 12.5

As the scale factor is greater than 1, the kite is an enlargement of the scale diagram.

$k = 12.5$

k represents the scale factor for the enlargement.

Area of kite $= k^2$(Area of scale diagram)
Area of kite $= (12.5)^2(20\text{ cm}^2)$
Area of kite $= 3125\text{ cm}^2$

I know that the scale diagram and the actual kite are similar shapes. This means that the area of the actual kite can be determined by multiplying the area of the scale diagram by the square of the scale factor.

I will need at least 3125 cm^2 of fabric for my kite.

This amount of fabric will cover the frame exactly. I'll need more than this amount, since I'll have to sew the fabric to the frame.

Hank's Solution: Reasoning about scale as a reduction

Let k represent the scale factor.

$$k = \frac{2}{25}$$

The scale diagram is a reduction of the kite since the scale factor is less than 1.

$$\text{Area of scale diagram} = k^2(\text{Area of kite})$$

$$\frac{\text{Area of scale diagram}}{\text{Area of kite}} = k^2$$

Since the scale diagram and the actual kite are similar shapes, the area of the scale diagram equals the product of the square of the scale factor k and the area of the actual kite.

Let x represent the area of the kite.

I substituted the information I knew into the equation.

$$\frac{20\text{ cm}^2}{x} = \left(\frac{2}{25}\right)^2$$

$$\frac{20\text{ cm}^2}{x} = \frac{4}{625}$$

$$12\,500\text{ cm}^2 = 4x$$

$$3125\text{c m}^2 = x$$

Jasmine will need at least 3125 cm^2 of fabric for her kite.

This amount of fabric will cover the frame exactly. She will need a little more than this amount, so that she can sew the fabric to the frame.

Your Turn

If the scale diagram for the kite had been drawn using a scale ratio of 1 : 20, and the area of the scale diagram had been 30 cm^2, how much fabric would Jasmine have needed for her kite?

EXAMPLE 2 Reasoning about scale factor and area to determine dimensions

Jim's laptop has a monitor with the dimensions 9 in. by 12 in. The image on his laptop is projected onto the screen of a whiteboard. According to the documentation for the whiteboard, its screen area is 2836.6875 in.^{2}.

a) The image on the whiteboard is similar to the image on the laptop. Determine the scale factor used to project the images on the laptop to the whiteboard.

b) Determine the dimensions of the whiteboard.

Rani's Solution

a) Area of monitor $= lw$
Area of monitor $= (9 \text{ in.})(12 \text{ in.})$
Area of monitor $= 108 \text{ in.}^2$
Let k represent the scale factor.
Area of whiteboard $= k^2$(Area of monitor)
$2836.6875\text{i n.}^2 = k^2(108 \text{ in.}^2)$
$\frac{2836.6875 \cancel{\text{in.}^2}}{108 \cancel{\text{in.}^2}} = k^2$
$26.265... = k^2$
$\sqrt{26.265...} = k$
$5.125 = k$

The laptop's monitor is a rectangle, so I determined its area by multiplying its length, l, and width, w.

The image on the laptop and the image on the whiteboard are similar rectangles. This means that the area of the image on the whiteboard is equal to the square of the scale factor times the area of the image on the laptop.

Since the image on the whiteboard is larger than the original, I know that $k > 1$. A scale factor of 5.125 makes sense.

The dimensions of the image on the whiteboard are an enlargement of the dimensions of the image on Jim's laptop by a factor of 5.125.

b) Let x represent the length of the whiteboard.
$x = (12 \text{ in.})(5.125)$
$x = 61.5 \text{ in.}$
Let y represent the width of the whiteboard.
$y = (9 \text{ in.})(5.125)$
$y = 46.125 \text{ in.}$

To determine the dimensions of the whiteboard, I multiplied the length and width of the laptop's monitor by the scale factor.

The whiteboard is about 61 in. long by 46 in. wide.

Your Turn

A circular icon on Jim's laptop has a diameter of 2 cm. Calculate the area of this icon on the whiteboard.

In Summary

Key Idea

- If two 2-D shapes are similar and their dimensions are related by a scale factor k, then the relationship between the area of the similar shape and the area of the original shape can be expressed as:

$$\text{Area of similar 2-D shape} = k^2(\text{Area of original shape})$$

Need to Know

- If the area of a similar 2-D shape and the area of the original shape are known, then the scale factor, k, can be determined using the formula

$$k^2 = \frac{\text{Area of similar 2-D shape}}{\text{Area of original shape}}$$

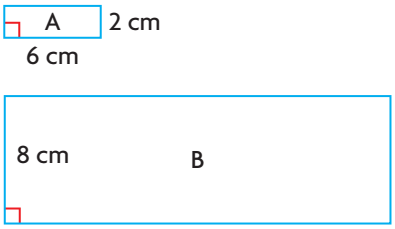
CHECK Your Understanding

1. Two similar rectangles, A and B, are shown to the right.

a) Determine the scale factor that produced the enlargement from rectangle A to rectangle B.

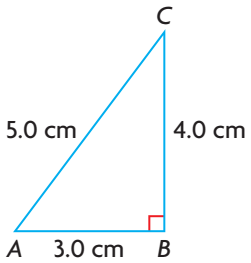
b) Determine the areas of rectangle A and rectangle B.

c) How many rectangles congruent to rectangle A would fit in rectangle B?



2. The table below gives data for enlargements and reductions of the triangle shown to the right. Complete the table.

Length of Base (cm)	Height of Triangle (cm)	Scale Factor	Area (cm^2)	Area of scaled triangle / Area of original triangle
3.0	4.0	1	6.0	1
		3		
1.5				
			600.0	
		25%		

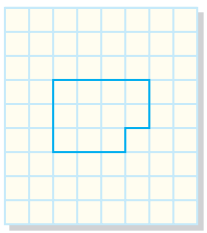


PRACTISING

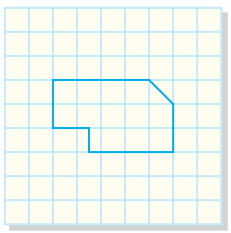
3. The parallelogram shown to the right has an area of 42 cm^2. It is going to be enlarged by a scale factor of 5. Determine the area of the enlarged parallelogram.

4. Determine the area of each figure after it is enlarged by a scale factor of 2.

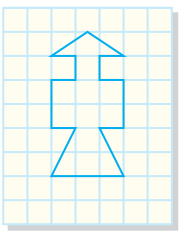
a)



b)

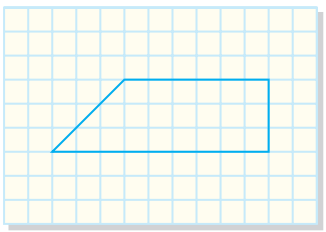


c)

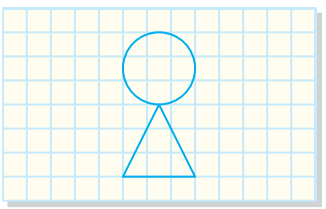


5. Determine the area of each figure, to the nearest tenth of a square unit, after it is reduced by a scale factor of $\frac{1}{3}$.

a)



b)



6. Tammy downloaded a photograph, which measured 4 in. by 6 in., from her camera to her laptop. Then Tammy used a software program to enlarge the dimensions of the photograph by 150% so that it would fit in a frame she already had.

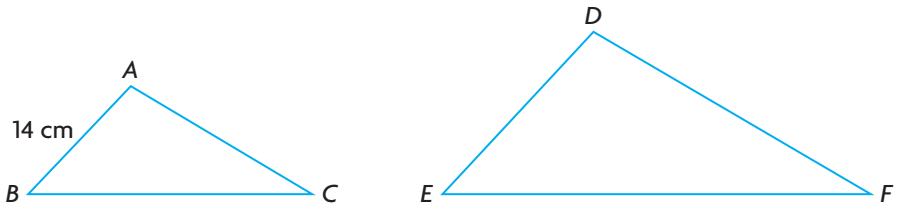
a) What are the inside dimensions of the frame she already had?

b) By what percent was the area of the photograph increased in the enlarging process?

c) Explain how you could determine the area of the enlarged photograph using two different strategies.

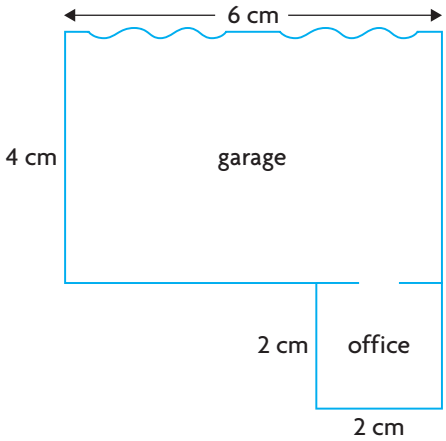
7. Stop signs on city, town, and rural roads are regular octagons. Describe how you would create a similar stop sign that is quadruple the area of a typical stop sign for increased visibility on a two-lane highway.

8. $\triangle ABC$ and $\triangle DEF$ are similar triangles. The sum of the lengths of AB and DE is 35 cm. The area of $\triangle DEF$ is 144 cm^2.



a) Determine the scale factor that relates $\triangle ABC$ to $\triangle DEF$.

b) Determine the area of $\triangle ABC$.



9. The sketch to the left of a service garage and an attached office was drawn using a scale ratio of 1 : 500. On this diagram, the area of the garage is 24 cm^2 and the area of the office is 4 cm^2. Determine the area of the actual garage and the actual office in square metres.

10. A rectangular display, with the dimensions 2 m by 3 m, is located in the lobby of city hall to show the citizens the layout for the new People's Park. The display was created using a scale ratio of 1 : 120.

a) The parks department estimates that the city spends \$0.75/m^2 to maintain a park from spring through fall. Estimate the cost to maintain People's Park.

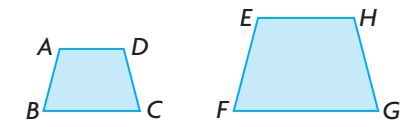
b) A rectangular model, with the same dimensions, was used to represent Meadow Park. The scale ratio used was 1 : 250. Estimate the cost to maintain Meadow Park.

11. A gymnasium wall is 20 ft high and 120 ft long. Peggy has been asked to paint a mural on the wall. The mural must be $\frac{1}{4}$ the area of the wall and the mural and wall must be similar. The mural must also be centred on the wall. Draw a scale diagram that shows the dimensions of the wall, the dimensions of the mural, and where the mural should be placed.

12. The scale ratio for two similar rectangles is 1 : 2. The sum of their areas is 40 cm^2. Determine the area of each rectangle.

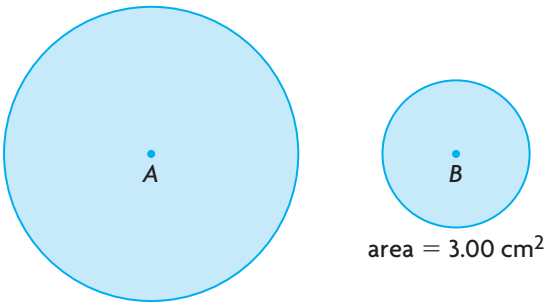
13. Determine the scale factor that relates each pair of similar shapes.

a)



area = 5.00 cm^2 area = 11.25 cm^2

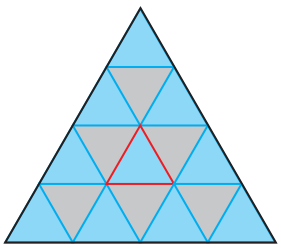
b)



area = 3.00 cm^2

area = 12.00 cm^2

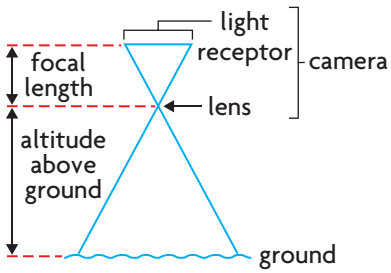
14. In the diagram to the right, the large triangle, outlined in black, is an enlargement of the small triangle, outlined in red. The small triangle is congruent to the other small triangles, which are equilateral and have side lengths of 1 unit.



a) Determine the value of the scale factor, k.

b) Explain how k relates the perimeters and areas of the large and small triangles.

15. Aerial photographs are often used to show parcels of land that are for sale. The camera used to take the photograph below had a focal length of 0.152 m. The altitude of the airplane was 7600 m when the photograph was taken. The ratio of the camera's focal length to the airplane's altitude is the scale factor for the photograph.



a) Determine the scale of the aerial photograph.

b) Determine the area of the parcel of land shown in the photograph in hectares. The conversion rate is 1 ha per 10 000 m^2.

c) Determine the value of the parcel of land, if it sells for $375/ha.

16. You would like to renovate the kitchen in your home, and you need to create floor plans of the renovations for the contractor.

a) Measure and record the dimensions of your kitchen. Include doors and measurements of any counters, appliances, or furniture that take up floor space.

b) On one piece of paper, draw two scale diagrams: the first diagram showing the existing kitchen and the second diagram showing how you would like the kitchen to appear after the renovation.

c) Compare the area of the open floor space in the two versions of the kitchen. Which version is more spacious?

Closing

17. Explain the difference between the following processes:

A. Reduce a 2-D shape by a scale factor of $\frac{1}{2}$.

B. Divide the area of the same 2-D shape by 2.

Use examples to support your explanation.

Extending

18. A polygon has its dimensions increased by 180% to create a similar polygon. The dimensions of the new polygon are then reduced by 50% to create a third polygon. What percent of the area of the original polygon is the area of the third polygon?

19. A company that manufactures cardboard boxes currently makes boxes with dimensions of 12 in. by 16 in. by 12 in. The company plans to make a new, larger box by increasing the current dimensions by 150%. Cardboard costs $0.05/sq ft. How much more will it cost to make the larger box?