7.8 Solving Problems Using Quadratic Models

GOAL

Analyze and solve problems that involve quadratic functions and equations.

YOU WILL NEED

- graphing technology

LEARN ABOUT the Math

The engineers who designed the Coal River Bridge on the Alaska Highway in British Columbia used a supporting arch with twin metal arcs.

The function that describes the arch is

$$h(x) = -0.005\,061x^2 + 0.499\,015x$$

where $h(x)$ is the height, in metres, of the arch above the ice at any distance, x, in metres, from one end of the bridge.

EXPLORE...

- A right triangle has sides of length x, $2x + 4$, and $3x - 4$. Write a quadratic equation to determine the value of x. Is there more than one solution?

? How can you use the width of the arch to determine the height of the bridge?

EXAMPLE 1 Solving a problem by factoring a quadratic equation

Determine the distance between the bases of the arch. Then determine the maximum height of the arch, to the nearest tenth of a metre.

Morgan's Solution

The coordinates of the maximum are (x, y), where x is halfway between the two bases of the arch and y is the height of the arch.

I reasoned that the bridge is symmetrical and resting on the vertex of the arch.

$$h(x) = -0.005\,061x^2 + 0.499\,015x$$
$$0 = -0.005\,061x^2 + 0.499\,015x$$

I wrote an equation to determine the x-coordinates of the bases of the arch. The height at each base is 0 m, so the value of $h(x)$ at these points is 0.

$0 = -5.061x^2 + 499.015x$
$0 = x(-5.061x + 499.015)$

I multiplied both sides by 1000 and factored the equation.

$x = 0$ or $-5.061x + 499.015 = 0$
$-5.061x = -499.015$
$x = 98.600...$

I solved the equation.

One base is at 0 m, and the other is at 98.600... m. The width of the arch is 98.600... m.

The width of the arch is the distance between the two bases.

Equation of axis of symmetry:

$$x = \frac{0 + 98.600...}{2}$$

$x = 49.300...$
The x-coordinate of the vertex is 49.300...

The function is quadratic, so the arch is a parabola with an axis of symmetry that passes through the vertex.

$h(x) = -0.005\ 061x^2 + 0.499\ 015x$
For $x = 49.300...$,
$y = -0.005\ 061\ (49.300...)^2 + 0.499\ 015\ (49.300...)$
$y = -12.300... + 24.601...$
$y = 12.300...$ m

The height of the arch is the y-coordinate of the vertex of the parabola.

The distance between the bases of the bridge is 98.6 m.
The height of the arch above the ice is 12.3 m.

Reflecting

A. How did determining the x-coordinates of the bases of the arch help Morgan determine the height of the arch?

B. What reasoning might have led Morgan to multiply both sides of the equation by 1000?

C. How did Morgan know that the equation $0 = -5.061x^2 + 499.015x$ could be factored?

D. How else could Morgan have solved her quadratic equation?

APPLY the Math

EXAMPLE 2 Solving a number problem by graphing

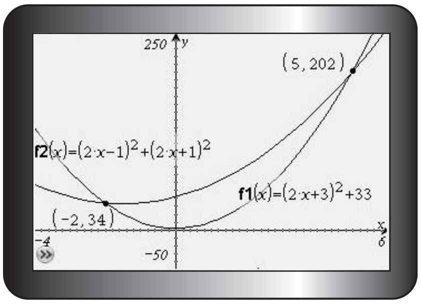
Determine three consecutive odd integers, if the square of the largest integer is 33 less than the sum of the squares of the two smaller integers.

Hailey's Solution

Let the three integers be $2x - 1$, $2x + 1$, and $2x + 3$.

$(2x + 3)^2 + 33 = (2x - 1)^2 + (2x + 1)^2$

Odd numbers are not divisible by 2. According to the problem, if I add 33 to the square of the largest number, my result will equal the sum of the squares of the two smaller numbers.



I graphed both sides of my equation and determined the points of intersection.

The points of intersection are $(-2, 34)$ and $(5, 202)$. The two possible values of x are -2 and 5.

If $x = -2$,

$2x - 1$	$2x + 1$	$2x + 3$
$2(-2) - 1$	$2(-2) + 1$	$2(-2) + 3$
-5	-3	-1

The integers are -5, -3, and -1.

I determined three consecutive odd integers for each value of x.

If $x = 5$,

$2x - 1$	$2x + 1$	$2x + 3$
$2(5) - 1$	$2(5) + 1$	$2(5) + 3$
9	11	13

The integers are 9, 11, and 13.
The consecutive odd integers could be -5, -3, and -1, or they could be 9, 11, and 13.

My answers seem reasonable. $13^2 = 169$ and this is 33 less than $9^2 + 11^2$, which is 202. $(-1)^2 = 1$ and this is 33 less than $(-5)^2 + (-3)^3$, which is 34.

Your Turn

Why was Hailey's method better for solving the problem than simply guessing and testing numbers?

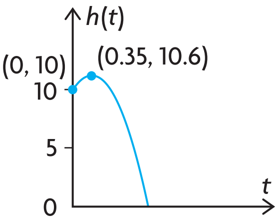
EXAMPLE 3 Solving a problem by creating a quadratic model

Synchronized divers perform matching dives from opposite sides of a platform that is 10 m high. If two divers reached their maximum height of 0.6 m above the platform after 0.35 s, how long did it take them to reach the water?

Canadians Émilie Heymans and Blythe Hartley won bronze medals at the 2004 Olympic Games.

Oliver's Solution

Let t represent the time in seconds.
Let $h(t)$ represent the height in metres over time.



I sketched a graph to show how the divers' height changed as time passed. I knew that the vertex of the parabola was (0.35, 10.6) because the maximum height of 10.6 m (0.6 m above the 10 m platform) was attained after 0.35 s.

$h(t) = a(t - 0.35)^2 + 10.6$

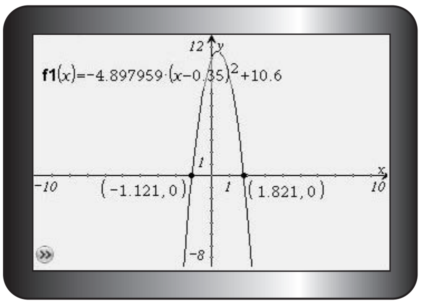
I wrote a quadratic function in vertex form.

$$10 = a(0 - 0.35)^2 + 10.6$$
$$10 = a(-0.35)^2 + 10.6$$
$$10 = 0.1225a + 10.6$$
$$-0.6 = 0.1225a$$
$$-4.897... = a$$

The platform is 10 m high. Therefore, when $t = 0$, $h(t) = 10$. I substituted these values into my equation and solved for a.

$f(x) = -4.897...(x - 0.35)^2 + 10.6$

I wrote a function to represent the dive. I knew that the height would be 0 when the divers hit the water.



I graphed my function and determined the x-intercepts.

The zeros of my function are -1.121 and 1.821.
The solution -1.121 s is inadmissible.

Time cannot be negative in this situation.

The divers reached the water after about 1.821 s.

Your Turn

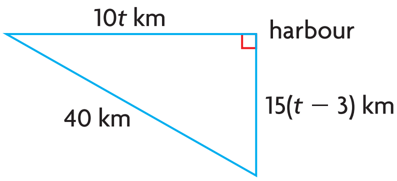
How does Oliver's first graph show that there is only one solution to the problem?

EXAMPLE 4 Visualizing a quadratic relationship

At noon, a sailboat leaves a harbour on Vancouver Island and travels due west at 10 km/h. Three hours later, another sailboat leaves the same harbour and travels due south at 15 km/h. At what time, to the nearest minute, will the sailboats be 40 km apart?

Nikki's Solution

Let t be the number of hours it will take for the sailboats to be 40 km apart.



I drew a diagram to show the paths of the two sailboats. They are sailing at right angles to each other.

The first boat travels t hours at 10 km/h.

The second boat leaves 3 h later, so it travels for $t - 3$ h at 15 km/h.

$$(10t)^2 + [15(t - 3)]^2 = 40^2$$

I used the Pythagorean theorem to write an equation that relates the distances travelled to the 40 km distance between them.

$$(10t)^2 + (15t - 45)^2 = 40^2$$
$$100t^2 + 225t^2 - 1350t + 2025 = 1600$$
$$325t^2 - 1350t + 425 = 0$$
$$13t^2 - 54t + 17 = 0$$

I simplified my quadratic equation and wrote it in standard form. Then I divided both sides by 25.

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$t = \frac{-(-54) \pm \sqrt{(-54)^2 - 4(13)(17)}}{2(13)}$$
$$t = \frac{54 \pm \sqrt{2032}}{26}$$
$$t = 0.343\ldots \quad \text{or} \quad t = 3.810\ldots$$

I used the quadratic formula to solve for t, the number of hours that it will take for the boats to be 40 km apart.

The solution 0.343... h is inadmissible.
(0.81... h)(60 min/1 h) = 48.6 min
The boats will be 40 km apart at 3:49 p.m.

The boats could not be 40 km apart after 0.343... h, because the second boat has not yet left the harbour and the first boat is less than 10 km out.

Your Turn

a) Tomas solved the same problem. However, he used t to represent the time for the second boat's journey. How would the labels on Tomas's diagram be different from the labels on Nikki's diagram?

b) Use Tomas's method to solve the problem.

In Summary

Key Ideas

- A function, a graph, or a table of values can represent a relation. Use the form that is most helpful for the context of the problem.
- Depending on the information that is given in a problem, you can use a quadratic function in vertex form or in standard form to model the situation.

Need to Know

- A problem may have only one admissible solution, even though the quadratic equation that is used to represent the problem has two real solutions. When you solve a quadratic equation, verify that your solutions make sense in the context of the problem.

CHECK *Your Understanding*

1. The engineers who built the Coal River Bridge on the Alaska Highway in British Columbia used scaffolding during construction. At one point, scaffolding that was 9 m tall was placed under the arch. The arch is modelled by the function

$$h(x) = -0.005\,061x^2 + 0.499\,015x$$

a) Describe a strategy you could use to determine the minimum distance of this scaffolding from each base of the arch.
b) Use your strategy from part a) to solve the problem.
c) Compare your strategy and solutions with a classmate's strategy and solutions. What other strategies could you have used?

PRACTISING

Communication *Tip*

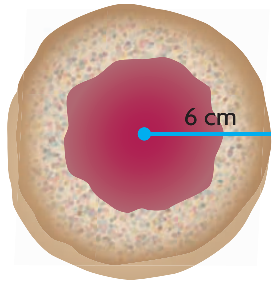
The formula $V = \pi r^2 h$ can also be written as a quadratic function:

$f(r) = \pi r^2 h - V$

2. A company manufactures aluminum cans. One customer places an order for cans that must be 18 cm high, with a volume of 1150 cm^3.
a) Use the formula $V = \pi r^2 h$ to determine the radius that the company should use to manufacture these cans.
b) Graph the function that corresponds to $0 = \pi r^2 h - V$ to determine the radius.
c) Which method do you prefer? Explain why.

3. The sum of two numbers is 11. Their product is −152. What are the numbers?

4. A doughnut store sells doughnuts with jam centres. The baker wants the area of the jam to be about equal to the area of the cake part of the doughnut, as seen from the top. The outer radius of a whole doughnut is 6 cm. Determine the radius of the jam centre.



5. Duncan dives with a junior swim club. In a dive off a 7.5 m platform, he reaches a maximum height of 7.94 m after 0.30 s. How long does it take him to reach the water?

6. A jet skier leaves a dock at 8 a.m. and travels due west at 36 km/h. A second jet skier leaves the same dock 10 min later and travels due south at 44 km/h. At what time of day, to the nearest minute, will the two jet skis be 20 km apart?

7. Alexis sells chocolate mousse tortes for $25. At this price, she can sell 200 tortes every week. She wants to increase her earnings, but, from her research, she knows that she will sell 5 fewer tortes per week for each price increase of $1.
   a) What function, $E(x)$, can be used to model Alexis's earnings, if x represents the price increase in dollars?
   b) What higher price would let Alexis earn the same amount of money she earns now?
   c) What should Alexis charge for her tortes if she wants to earn the maximum amount of money?

8. Two consecutive integers are squared. The sum of these squares is 365. What are the integers?

9. Brianne is a photographer in southern Alberta. She is assembling a display of photographs of endangered local wildlife. She wants each photograph in her display to be square, and she wants the matte surrounding each photograph to be 6 cm wide. She also wants the area of the matte to be equal to the area of the photograph itself. What should the dimensions of each photograph be, to the nearest tenth of a centimetre?

Closing

10. Quadratic equations that describe problem situations are sometimes complicated. What are some methods you can use to simplify these equations and make them easier to solve?

Extending

11. Aldrin and Jan are standing at the edge of a huge field. At 2:00 p.m., Aldrin begins to walk along a straight path at a speed of 3 km/h. Two hours later, Jan takes a straight path at a 60° angle to Aldrin's path, walking at 5 km/h. At what time will the two friends be 13 km apart?

12. Frances is an artist. She wants the area of the matte around her new painting to be twice the area of the painting itself. The matte that she wants to use is available in only one width. The outside dimensions of the same matte around another painting are 80 cm by 60 cm. What is the width of the matte?

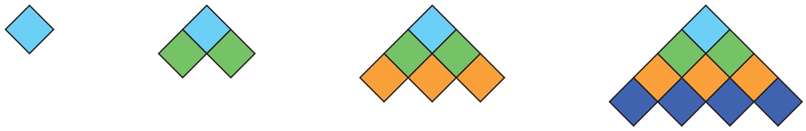
Applying Problem-Solving Strategies

Determining Quadratic Patterns

Many geometric patterns have connections to algebra. Examining a pattern can help you develop a formula that describes the general rule for the pattern.

The Puzzle

This pattern grows as a new row of tiles is added to each figure.



How many tiles would you need to construct a figure with 12 rows?

The Strategy

A. Copy this table.

Number of Rows	0	1	2	3	4			
Number of Tiles in Bottom Row	0	1	2	3				
Total Number of Tiles	0	1	3	6				

B. Complete your table for the next three figures in the pattern above.

C. Explain how you would determine the total number of tiles in figures with 8, 9, and 10 rows.

D. Write a quadratic equation that gives the total number of tiles in a figure with any number of rows.

E. Test your equation by using it to determine the total number of tiles in a figure with 12 rows. Check your answer by extending your table.

F. When you developed your equation for the pattern, did you use inductive or deductive reasoning? Explain.