7.7 Solving Quadratic Equations Using the Quadratic Formula

YOU WILL NEED

- graphing technology

GOAL

Use the quadratic formula to determine the roots of a quadratic equation.

EXPLORE...

- Kyle was given the following function:

  $y = 2x^2 + 12x - 14$

  He wrote it in vertex form:

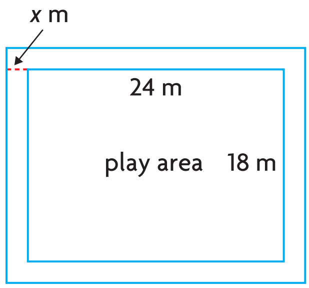
  $y = 2(x + 3)^2 - 32$

  How can you use the vertex form to solve this equation?

  $2x^2 + 12x - 14 = 0$

LEARN ABOUT the Math

Ian has been hired to lay a path of uniform width around a rectangular play area, using crushed rock. He has enough crushed rock to cover 145 m^2.



? If Ian uses all the crushed rock, how wide will the path be?

EXAMPLE 1 Using the quadratic formula to solve a quadratic equation

Determine the width of the path that will result in an area of 145 m^2.

Alima's Solution

Area of border = Total area − Play area

The play area is a constant, (length)(width) or (24 m)(18 m) or 432 m^2.

The total area of the playground, P, can be represented as

$P = (\text{length})(\text{width})$

$P = (2x + 24)(2x + 18)$

I wrote a function that describes how the area of the path, A square metres, changes as the width of the path, x metres, changes.

The area of the path, $A(x)$, can be represented as

$A(x) = (2x + 24)(2x + 18) - 432$

$A(x) = 4x^2 + 84x + 432 - 432$

$A(x) = 4x^2 + 84x$

$145 = 4x^2 + 84x$

I substituted the area of 145 m^2 for $A(x)$.

$4x^2 + 84x - 145 = 0$
$a = 4$, $b = 84$, and $c = -145$

I rewrote the equation in standard form:

$ax^2 + bx + c = 0$

Then I determined the values of the coefficients a, b, and c.

Quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-84 \pm \sqrt{84^2 - 4(4)(-145)}}{2(4)}$$

$$x = \frac{-84 \pm \sqrt{9376}}{8}$$

$$x = \frac{-84 + \sqrt{9376}}{8} \quad \text{or}$$

$$x = \frac{-84 - \sqrt{9376}}{8}$$

$$x = 1.603 \ldots \quad \text{or} \quad x = -22.603 \ldots$$

The **quadratic formula** can be used to solve any quadratic equation. I wrote the quadratic formula and then substituted the values of a, b, and c from my equation into the formula.

I simplified the right side.

I separated the quadratic expression into two solutions.

quadratic formula
A formula for determining the roots of a quadratic equation in the form $ax^2 + bx + c = 0$, where $a \neq 0$; the quadratic formula is written using the coefficients of the variables and the constant in the quadratic equation that is being solved:

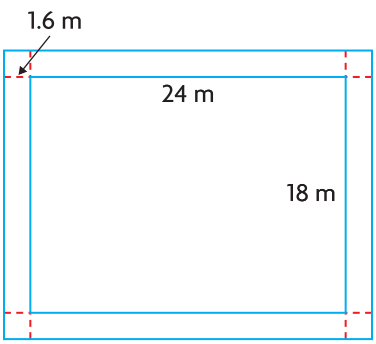
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This formula is derived from $ax^2 + bx + c = 0$ by isolating x.

The solution -22.603 is inadmissible.

I knew that the width of the path couldn't be negative, so $-22.603 \ldots$ is an **inadmissible solution**.

inadmissible solution
A root of a quadratic equation that does not lead to a solution that satisfies the original problem.



I sketched the path and verified my solution by determining the area of the path. To do this, I added the areas of all the rectangles that make up the path.

$18(1.603 \ldots) = 28.866 \ldots \text{ m}^2$
$24(1.603 \ldots) = 38.489 \ldots \text{ m}^2$
$(1.603 \ldots)(1.603 \ldots) = 2.571 \ldots \text{ m}^2$

Area of path $= 2(28.866 \ldots) + 2(38.489 \ldots) + 4(2.571 \ldots)$
Area of path $= 144.999 \ldots \text{ m}^2$

The total area is very close to 145 m^2.
The path should be about 1.6 m wide.

Reflecting

A. Why did Alima need to write her equation in standard form?

B. Which part of the quadratic formula shows that there are two possible solutions?

C. Why did Alima decide not to use the negative solution?

D. In this chapter, you have learned three methods for solving quadratic equations: graphing, factoring, and using the quadratic formula. What are some advantages and disadvantages of each method?

APPLY the Math

EXAMPLE 2 Connecting the quadratic formula to factoring

Solve the following equation:

$$6x^2 - 3 = 7x$$

Adrianne's Solution

$$6x^2 - 3 = 7x$$
$$6x^2 - 7x - 3 = 0$$
$$a = 6,\ b = -7, \text{ and } c = -3$$

First, I rewrote the equation in standard form to determine the values of *a*, *b*, and *c*.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(6)(-3)}}{2(6)}$$

$$x = \frac{7 \pm \sqrt{121}}{12}$$

$$x = \frac{7 \pm 11}{12}$$

I wrote the quadratic formula and substituted the values of *a*, *b*, and *c*.

I simplified the right side. I realized that 121 is a perfect square.

$$x = \frac{18}{12} \quad \text{or} \quad x = \frac{-4}{12}$$

$$x = \frac{3}{2} \quad \text{or} \quad x = \frac{-1}{3}$$

I determined the two solutions.

Verify:

$$6x^2 - 7x - 3 = 0$$
$$(3x + 1)(2x - 3) = 0$$

$$3x + 1 = 0 \quad \text{or} \quad 2x - 3 = 0$$
$$3x = -1 \qquad 2x = 3$$
$$x = \frac{-1}{3} \qquad x = \frac{3}{2}$$

The solutions match those I got using the quadratic formula.

If the radicand in the quadratic formula is a perfect square, then the original equation can be factored. I decided to verify my solution by factoring the original equation.

Your Turn

Sandy was given the following equation:

$$12x^2 - 47x + 45 = 0$$

She used the quadratic formula to solve it.

$$x = \frac{-(-47) \pm \sqrt{(-47)^2 - 4(12)(45)}}{2(12)}$$

$$x = \frac{47 \pm \sqrt{49}}{24}$$

$$x = 2\frac{1}{4} \quad \text{or} \quad x = \frac{5}{3}$$

Could Sandy use factoring to verify her solutions? Explain how you know.

EXAMPLE 3 Determining the exact solution to a quadratic equation

Solve this quadratic equation:

$$2x^2 + 8x - 5 = 0$$

State your answer as an exact value.

Quyen's Solution

$2x^2 + 8x - 5 = 0$

$a = 2$, $b = 8$, and $c = -5$

The equation was in standard form. I determined the values of *a*, *b*, and *c*.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(8) \pm \sqrt{8^2 - 4(2)(-5)}}{2(2)}$$

I wrote the quadratic formula and substituted the values of *a*, *b*, and *c*.

$$x = \frac{-8 \pm \sqrt{104}}{4}$$

$$x = \frac{-8 \pm \sqrt{4}\sqrt{26}}{4}$$

$$x = \frac{-8 \pm 2\sqrt{26}}{4}$$

$$x = \frac{-4 \pm \sqrt{26}}{2}$$

I simplified the expression.

I noticed that one factor of 104 is 4, which is a perfect square. I simplified the radical.

I simplified the fraction.

$$x = \frac{-4 + \sqrt{26}}{2} \quad \text{or} \quad x = \frac{-4 - \sqrt{26}}{2}$$

Another way to write my solution is to show two separate values.

Your Turn

Solve the following quadratic equation:

$$5x^2 - 10x + 3 = 0$$

State your answer as an exact value.

EXAMPLE 4 Solving a pricing problem

A store rents an average of 750 video games each month at the current rate of $4.50. The owners of the store want to raise the rental rate to increase the revenue to $7000 per month. However, for every $1 increase, they know that they will rent 30 fewer games each month. The following function relates the price increase, p, to the revenue, r:

$$(4.5 + p)(750 - 30p) = r$$

Can the owners increase the rental rate enough to generate revenue of $7000 per month?

Christa's Solution

$$(4.5 + p)(750 - 30p) = r$$
$$3375 + 615p - 30p^2 = r$$
$$3375 + 615p - 30p^2 = 7000$$
$$-30p^2 + 615p + 3375 = 7000$$
$$-30p^2 + 615p - 3625 = 0$$
$$\frac{-30p^2}{-5} + \frac{615p}{-5} - \frac{3625}{-5} = \frac{0}{-5}$$
$$6p^2 - 123p + 725 = 0$$

I simplified the function.

I substituted the revenue of $7000 for r and wrote the equation in standard form.

I divided each term by −5 to simplify the equation.

$$p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$p = \frac{-(-123) \pm \sqrt{(-123)^2 - 4(6)(725)}}{2(6)}$$
$$p = \frac{123 \pm \sqrt{-2271}}{12}$$

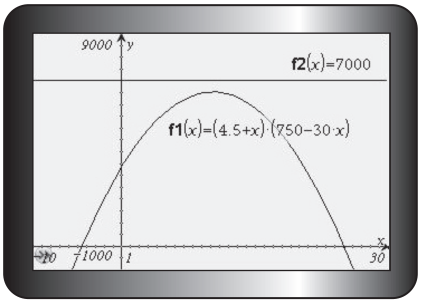
I didn't try to factor the equation since the numbers were large. I decided to use the quadratic formula.

I substituted the values of a, b, and c into the quadratic formula.

I simplified the right side.

$\sqrt{-2271}$ is not a real number, so there are no real solutions to this equation. It is not possible for the store to generate revenue of $7000 per month by increasing the rental rate.

I noticed that the radicand is negative.



To verify my answer, I graphed

$y = (4.5 + x)(750 - 30x)$ and $y = 7000$

There is no point of intersection.

Your Turn

Is it possible for the store to generate revenue of \$6500 per month by increasing the rental rate? Explain.

In Summary

Key Idea

- The roots of a quadratic equation in the form $ax^2 + bx + c = 0$, where $a \neq 0$, can be determined by using the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Need to Know

- The quadratic formula can be used to solve any quadratic equation, even if the equation is not factorable.
- If the radicand in the quadratic formula simplifies to a perfect square, then the equation can be solved by factoring.
- If the radicand in the quadratic formula simplifies to a negative number, then there is no real solution for the quadratic equation.

CHECK Your Understanding

1. Solve each equation using the quadratic formula. Verify by graphing.

a) $x^2 + 7x - 5 = 0$
b) $8x^2 + 35x + 12 = 0$
c) $2a^2 - 5a + 1 = 0$
d) $-20p^2 + 7p + 3 = 0$

2. Solve each equation using the quadratic formula.

a) $x^2 + 5x - 6 = 0$
b) $4x + 9x^2 = 0$
c) $25x^2 - 121 = 0$
d) $12x^2 - 17x - 40 = 0$

3. Solve each equation in question 2 by factoring. Which method did you prefer for each equation? Explain.

PRACTISING

4. Solve each quadratic equation.

a) $3x^2 + 5x = 9$ **c)** $6x - 3 = 2x^2$

b) $1.4x - 3.9x^2 = -2.7$ **d)** $x^2 + 1 = x$

5. The roots for the quadratic equation

$$1.44a^2 + 2.88a - 21.6 = 0$$

are $a = 3$ and $a = -5$. Verify these roots.

6. Solve each equation. State the solutions as exact values.

a) $3x^2 - 6x - 1 = 0$ **c)** $8x^2 + 8x - 1 = 0$

b) $x^2 + 8x + 3 = 0$ **d)** $9x^2 - 12x - 1 = 0$

7. A student council is holding a raffle to raise money for a charity fund drive. The profit function for the raffle is

$$p(c) = -25c^2 + 500c - 350$$

where $p(c)$ is the profit and c is the price of each ticket, both in dollars.

a) What ticket price will result in the student council breaking even on the raffle?

b) What ticket price will raise the most money for the school's donation to charity?

Akpatok Island gets its name from the word *Akpat*, the Innu name for the birds that live on the cliffs.

8. Akpatok Island in Nunavut is surrounded by steep cliffs along the coast. The cliffs range in height from about 125 m to about 250 m.

a) Suppose that someone accidentally dislodged a stone from a 125 m cliff. The height of the stone, $h(t)$, in metres, after t seconds can be represented by the following function:

$$h(t) = -4.9t^2 + 4t + 125$$

How long would it take the stone dislodged from this height to reach the water below?

b) Predict how much longer it would take for the stone to reach the water if it fell from a height of 250 m. Discuss this with a partner.

c) The height of a stone, $h(t)$, in metres, falling from a 250 m cliff over time, t, in seconds, can be modelled by this function:

$$h(t) = -4.9t^2 + 4t + 250$$

Determine how long it would take the stone to reach the water.

d) How close was your prediction to your solution?

9. Keisha and Savannah used different methods to solve this equation:

$$116.64z^2 + 174.96z + 65.61 = 0$$

a) Could one of these students have used factoring? Explain.

b) Solve the equation using the method of your choice.

c) Which method did you use? Why?

10. The Moon's gravity affects the way that objects travel when they are thrown on the Moon. Suppose that you threw a ball upward from the top of a lunar module, 5.5 m high. The height of the ball, $h(t)$, in metres, over time, t, in seconds could be modelled by this function:

$$h(t) = -0.81t^2 + 5t + 6.5$$

a) How long would it take for the ball to hit the surface of the Moon?

b) If you threw the same ball from a model of the lunar module on Earth, the height of the ball could be modelled by this function:

$$h(t) = -4.9t^2 + 5t + 6.5$$

Compare the time that the ball would be in flight on Earth with the time that the ball would be in flight on the Moon.

Six lunar modules landed on the Moon from 1969 to 1972.

11. A landscaper is designing a rectangular garden, which will be 5.00 m wide by 6.25 m long. She has enough crushed rock to cover an area of 6.0 m^2 and wants to make a uniform border around the garden. How wide should the border be, if she wants to use all the crushed rock?

Closing

12. Discuss the quadratic formula with a partner. Make a list of everything you have both learned, from your work in this lesson, about using the quadratic formula to solve quadratic equations.

Extending

13. The two roots of any quadratic equation are

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

a) Determine the sum of the roots of any quadratic equation.

b) Determine the product of the roots of any quadratic equation.

c) Solve the following quadratic equation:

$$10x^2 - 13x + 4 = 0$$

Determine the sum and the product of its roots.

d) Determine the sum and the product of the roots of the quadratic equation in part c), using your formulas from parts a) and b). Do your answers match your answers from part c)?

e) Determine the sum and the product of the solutions to questions 1 d), 2 a), 5, and 7.

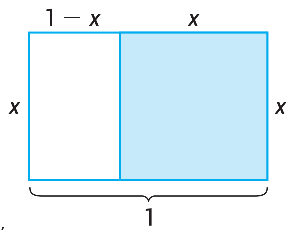
f) How could you use your formulas from parts a) and b) to check your solutions to any quadratic equation?

History | **Connection**

The Golden Ratio

The golden ratio has been discovered and rediscovered by many civilizations. Its uses in architecture include the Great Pyramids in Egypt and the Parthenon in Greece. The golden ratio is the ratio of length to width in a rectangle with special properties, called the golden rectangle. This rectangle appears often in art, architecture, and photography.

The Manitoba Legislative Building



If you section off a square inside a golden rectangle so that the side length of the square equals the width of the golden rectangle, you will create a smaller rectangle with the same length : width ratio. Mathematicians sometimes use the Greek letter *phi*, φ, to represent this ratio.

To determine the golden ratio, you need to know that the ratio of length to width in the original rectangle, $\frac{1}{x}$, is equal to the ratio of length to width in the smaller rectangle, $\frac{x}{1-x}$.

A. Solve the following equation for x to determine the width of a rectangle with length 1. Then determine $\frac{1}{x}$ to get the golden ratio, φ.

$$\frac{1}{x} = \frac{x}{1-x}$$

B. Work with a partner or group to find golden rectangles in the photograph of the Manitoba Legislative Building.

C. Find more golden rectangles in architecture, art, and nature. Present your findings to the class.