7.6 Vertex Form of a Quadratic Function

YOU WILL NEED

- graph paper and ruler
- graphing technology
- calculator

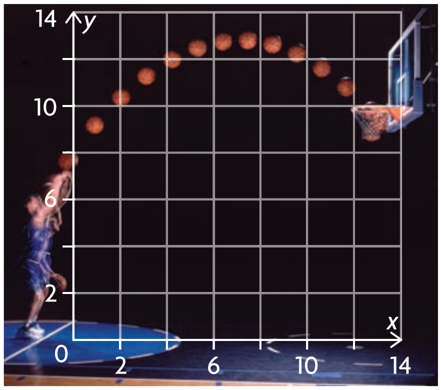
EXPLORE…

- Quadratic functions can be written in different forms. The basic quadratic function is $y = x^2$.
  Use a calculator to graph the following quadratic functions. Explain how the basic function is related to each function. Describe how the changes in the function affect the graph.
  a) $y = (x - 3)^2$
  b) $y = x^2 - 5$
  c) $y = (x + 1)^2 - 2$
  d) $y = (x + 4)^2 + 6$
  e) $y = -2(x + 1)^2 + 3$
  f) $y = 3(x - 2)^2 - 4$

GOAL

Graph a quadratic function in the form $y = a(x - h)^2 + k$, and relate the characteristics of the graph to its equation.

INVESTIGATE the Math



A high-school basketball coach brought in Judy, a trainer from one of the local college teams, to talk to the players about shot analysis. Judy demonstrated, using stroboscopic photographs, how shots can be analyzed and represented by quadratic functions. She used the following function to model a shot:

$$y = -0.1(x - 8)^2 + 13$$

In this function, x represents the horizontal distance, in feet, of the ball from the player and y represents the vertical height, in feet, of the ball above the floor.

Judy mentioned that once she had a quadratic equation in this form, she did not need the photographs. She could quickly sketch a graph of the path of the ball just by looking at the equation.

? How could Judy predict what the graph of the quadratic function would look like?

A. Graph the following function:

$$y = x^2$$

Change the graph by changing the coefficient of x^2. Try both positive and negative values. How do the parabolas change as you change this coefficient?

B. For each function you graphed in part A, determine the coordinates of the vertex and the equation of the axis of symmetry.

C. Graph this function:

$$y = x^2 + 1$$

Change the graph by changing the constant. Try both positive and negative values. How do the parabolas change as you change the constant? How do the coordinates of the vertex and the equation of the axis of symmetry change?

D. Graph this function:

$$y = (x - 1)^2$$

Change the graph by changing the constant. Try both positive and negative values. How do the parabolas change as you change the constant? How do the coordinates of the vertex and the equation of the axis of symmetry change?

E. The equation that Judy used was expressed in vertex form:

$$y = a(x - h)^2 + k$$

Make a conjecture about how the values of a, h, and k determine the characteristics of a parabola.

F. Test your conjecture by predicting the characteristics of the graph of the following function:

$$y = -0.1(x - 8)^2 + 13$$

Use your predictions to sketch a graph of the function.

G. Using a graphing calculator, graph the function from part F:

$$y = -0.1(x - 8)^2 + 13$$

How does your sketch compare with this graph? Are your predictions supported? Explain.

Communication *Tip*

A quadratic function is in vertex form when it is written in the form $y = a(x - h)^2 + k$

Reflecting

H. Does the value of a in a quadratic function always represent the same characteristic of the parabola, whether the function is written in standard form, factored form, or vertex form? Explain.

I. Neil claims that when you are given the vertex form of a quadratic function, you can determine the domain and range without having to graph the function. Do you agree or disagree? Explain.

J. Which form of the quadratic function—standard, factored, or vertex—would you prefer to start with, if you wanted to sketch the graph of the function? Explain.

APPLY the Math

EXAMPLE 1 Sketching the graph of a quadratic function given in vertex form

Sketch the graph of the following function:

$$f(x) = 2(x - 3)^2 - 4$$

State the domain and range of the function.

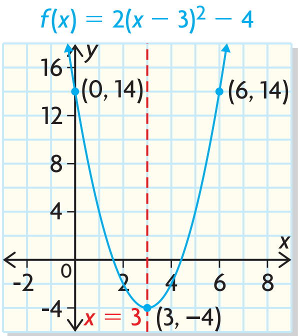
Samuel's Solution

$f(x) = 2(x - 3)^2 - 4$
Since $a > 0$, the parabola opens upward.
The vertex is at $(3, -4)$.
The equation of the axis of symmetry is $x = 3$.

The function was given in vertex form. I listed the characteristics of the parabola that I could determine from the equation.

$f(0) = 2(0 - 3)^2 - 4$
$f(0) = 2(-3)^2 - 4$
$f(0) = 2(9) - 4$
$f(0) = 18 - 4$
$f(0) = 14$
Point $(0, 14)$ is on the parabola.

To determine another point on the parabola, I substituted 0 for x.



I plotted the vertex and the point I had determined, (0, 14). Then I drew the axis of symmetry. I used symmetry to determine the point that is the same horizontal distance from (0, 14) to the axis of symmetry. This point is (6, 14). I connected all three points with a smooth curve.

Domain and range:
$\{(x, y) \mid x \in \mathrm{R}, y \geq -4, y \in \mathrm{R}\}$

Your Turn

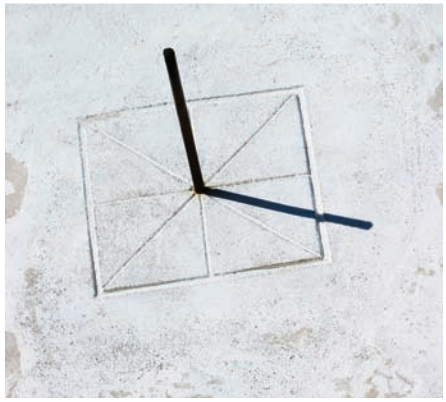
Sketch the graph of the following function:

$$f(x) = -\frac{1}{2}(x + 6)^2 + 1$$

State the domain and range of the function. Justify your decision.

EXAMPLE 2 Determining the equation of a parabola using its graph

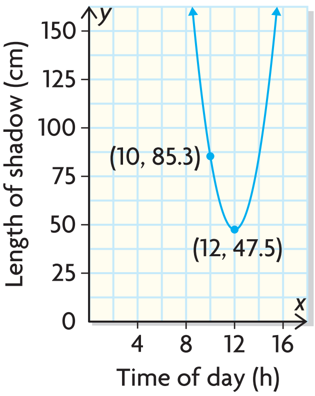
Liam measured the length of the shadow that was cast by a metre stick at 10 a.m. and at noon near his home in Saskatoon. Other students in his class also measured the shadow at different times during the day. They had read that, when graphed as shadow length versus time, the data should form a parabola with a minimum at noon, because the shadow is shortest at noon. Liam decided to try to predict the equation of the parabola, without the other students' data.



Determine the equation that represents the relationship between the time of day and the length of the shadow cast by a metre stick.

Liam's Solution

I have the points (10, 85.3) and (12, 47.5).



I measured the length of the shadow in centimetres. My measurements were 85.3 cm at 10 a.m. and 47.5 cm at noon.

I drew a sketch of a parabola using (12, 47.5) as the vertex, since the length of the shadow at noon should be the minimum value of the function.

$f(x) = a(x - h)^2 + k$
$f(x) = a(x - 12)^2 + 47.5$

I decided to use the vertex form of the quadratic function, since I already knew the values of *h* and *k* in this form.

Solving for a:
$85.3 = a(10 - 12)^2 + 47.5$
$85.3 = a(-2)^2 + 47.5$
$85.3 = 4a + 47.5$
$37.8 = 4a$
$9.45 = a$

I knew that (10, 85.3) is a point on the parabola. I substituted the coordinates of this point into the equation and then solved for *a*.

The function that represents the parabola is
$f(x) = 9.45(x - 12)^2 + 47.5$

The domain and range of this function depend on the hours of daylight, which depends on the time of year.

Your Turn

Donald, a classmate of Liam's, lives across the city. Donald measured the length of the shadow cast by a metre stick as 47.0 cm at noon and 198.2 cm at 4:00 p.m. Determine a quadratic function using Donald's data, and explain how his function is related to Liam's function.

EXAMPLE 3 Reasoning about the number of zeros that a quadratic function will have

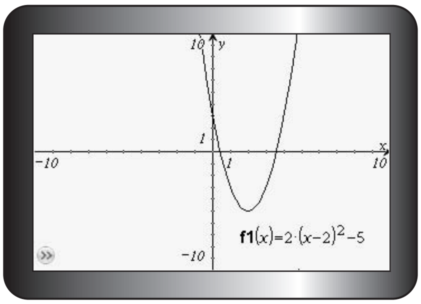
Randy claims that he can predict whether a quadratic function will have zero, one, or two zeros if the function is expressed in vertex form. How can you show that he is correct?

Eugene's Solution

$f(x) = 2(x - 2)^2 - 5$

Conjecture: two zeros

The vertex of the parabola that is defined by the function is at (2, -5), so the vertex is below the x-axis. The parabola must open upward because a is positive. Therefore, I should observe two x-intercepts when I graph the function.



To test my conjecture, I graphed the function on a calculator. I can see two x-intercepts on my graph, so the function has two zeros.

The graph supports my conjecture.

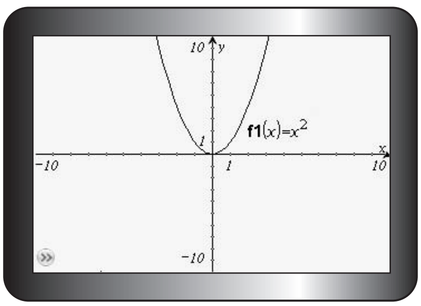
$f(x) = x^2$

$f(x) = (x - 0)^2 + 0$

Conjecture: one zero

I decided to use the basic quadratic function, since this provided me with a convenient location for the vertex, (0, 0).

Since the vertex is on the x-axis and the parabola opens up, this means that I should observe only one x-intercept when I graph the function.

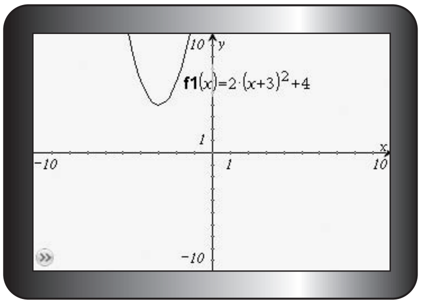


To test my conjecture, I graphed the function on a calculator. Based on my graph, I concluded that the function has only one zero.

The graph supports my conjecture.

$f(x) = 2(x + 3)^2 + 4$
Conjecture: no zeros

The vertex of the parabola that is defined by this function is at $(-3, 4)$, and the parabola opens upward. The vertex lies above the x-axis, so I should observe no x-intercepts when I graph the function.



To test my conjecture, I graphed the function on a calculator. I concluded that the function has no zeros.

The graph supports my conjecture.

Your Turn

a) Define three different quadratic functions, in vertex form, that open downward. One function should have two zeros, another should have one zero, and the third should have no zeros.

b) Explain how you were able to connect the number of zeros to each function.

EXAMPLE 4 Solving a problem that can be modelled by a quadratic function

A soccer ball is kicked from the ground. After 2 s, the ball reaches its maximum height of 20 m. It lands on the ground at 4 s.

a) Determine the quadratic function that models the height of the kick.

b) Determine any restrictions that must be placed on the domain and range of the function.

c) What was the height of the ball at 1 s? When was the ball at the same height on the way down?

Tia's Solution

a) Let x represent the elapsed time in seconds, and let y represent the height in metres.
$y = a(x - h)^2 + k$

Since I knew the maximum height and when it occurred, I also knew the coordinates of the vertex. I decided to use the vertex form to determine the equation.

The maximum height is 20 m at the elapsed time of 2 s.
Vertex:
$(x, y) = (2, 20)$

$y = a(x - 2)^2 + 20$

I substituted the known values.

Solving for a:

$$\begin{aligned} f(x) &= a(4 - 2)^2 + 20 \\ 0 &= a(2)^2 + 20 \\ 0 &= 4a + 20 \\ -20 &= 4a \\ -5 &= a \end{aligned}$$

To determine the value of a, I substituted the coordinates of the point that corresponds to the ball hitting the ground, (4, 0).

The following quadratic function models the height of the kick:
$f(x) = -5(x - 2)^2 + 20$

b) Time at beginning of kick:
$x = 0$

Time when ball hits ground:
$x = 4$

Domain: $\{x \mid 0 \leq x \leq 4, x \in \mathrm{R}\}$

At the beginning of the kick, the time is 0 *s*. When the ball lands, the time is 4 *s*. I can only use *x*-values in this interval. Time in seconds is continuous, so the set is real numbers.

Vertex: (2, 20)
Height of ball at beginning of kick: 0 m
Height of ball at vertex: 20 m

Range: $\{y \mid 0 \leq y \leq 20, y \in \mathrm{R}\}$

The ball starts on the ground, at a height of 0 m, and rises to its greatest height, 20 m. The ball is not below the ground at any point. Height in metres is continuous, so the set is real numbers.

c) $f(x) = -5(x - 2)^2 + 20$
$f(1) = -5(1 - 2)^2 + 20$
$f(1) = -5(-1)^2 + 20$
$f(1) = -5 + 20$
$f(1) = 15$

I used the vertex form of the quadratic function to determine the height of the ball at 1 s.

The ball was at a height of 15 m after 1 s.
This occurred as the ball was rising.

Equation of the axis of symmetry:
$x = 2$

Symmetry provides the point (3, 15).
The ball was also 15 m above the ground at 3 s.
This occurred as the ball was on its way down.

I knew that point (1, 15) is 1 unit to the left of the axis of symmetry of the parabola. The other point on the parabola, with height 15 m, should be 1 unit to the right of the axis of symmetry. This means that the *x*-coordinate of the point must be 3.

Your Turn

The goalkeeper kicked the soccer ball from the ground. It reached its maximum height of 24.2 m after 2.2 s. The ball was in the air for 4.4 s.

a) Define the quadratic function that models the height of the ball above the ground.
b) How is the equation for this function similar to the equation that Tia determined? Explain.
c) After 4 s, how high was the ball above the ground?

In Summary

Key Idea

- The vertex form of the equation of a quadratic function is written as follows:

$$y = a(x - h)^2 + k$$

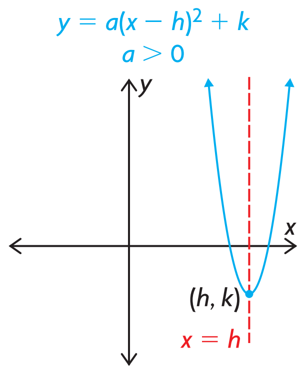
The graph of the function can be sketched more easily using this form.

Need to Know

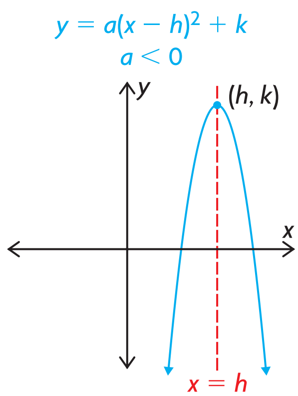
- A quadratic function that is written in vertex form,

$$y = a(x - h)^2 + k$$

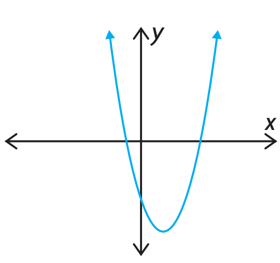
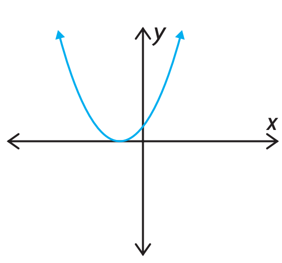
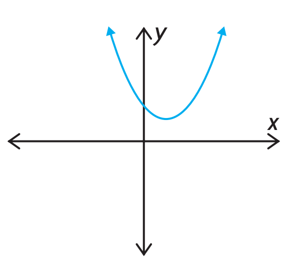
has the following characteristics:
  - The vertex of the parabola has the coordinates (h, k).
  - The equation of the axis of symmetry of the parabola is $x = h$.



  - The parabola opens upward when $a > 0$, and the function has a minimum value of k when $x = h$.
  - The parabola opens downward when $a < 0$, and the function has a maximum value of k when $x = h$.



- A parabola may have zero, one, or two x-intercepts, depending on the location of the vertex and the direction in which the parabola opens. By examining the vertex form of the quadratic function, it is possible to determine the number of zeros, and therefore the number of x-intercepts.

Two x-intercepts	One x-intercept	No x-intercepts
		

CHECK Your Understanding

1. For each quadratic function below, identify the following:

i) the direction in which the parabola opens

ii) the coordinates of the vertex

iii) the equation of the axis of symmetry

a) $f(x) = (x - 3)^2 + 7$

b) $m(x) = -2(x + 7)^2 - 3$

c) $g(x) = 7(x - 2)^2 - 9$

d) $n(x) = \frac{1}{2}(x + 1)^2 + 10$

e) $r(x) = -2x^2 + 5$

2. Predict which of the following functions have a maximum value and which have a minimum value. Also predict the number of x-intercepts that each function has. Test your predictions by sketching the graph of each function.

a) $f(x) = -x^2 + 3$

b) $q(x) = -(x + 2)^2 - 5$

c) $m(x) = (x + 4)^2 + 2$

d) $n(x) = (x - 3)^2 - 6$

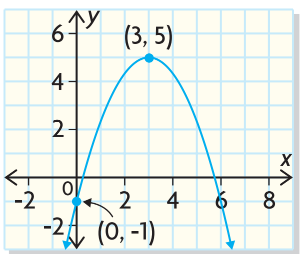
e) $r(x) = 2(x - 4)^2 + 2$

3. Determine the value of a, if point $(-1, 4)$ is on the quadratic function:

$$f(x) = a(x + 2)^2 + 7$$

PRACTISING

4. Which equation represents the graph? Justify your decision.



A. $y = -\frac{2}{3}x^2 + 5$

B. $y = -(x - 3)^2 + 5$

C. $y = -\frac{2}{3}(x - 3)^2 + 5$

D. $y = \frac{2}{3}(x - 3)^2 + 5$

5. Match each equation with its corresponding graph. Explain your reasoning.

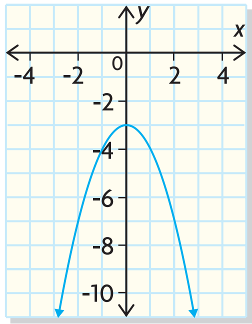
a) $y = (x - 3)^2$

b) $y = -(x + 4)^2 - 2$

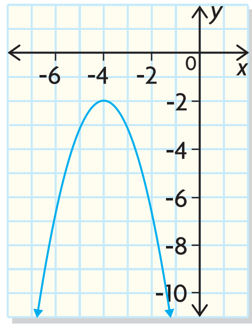
c) $y = -x^2 - 3$

d) $y = (x - 4)^2 + 2$

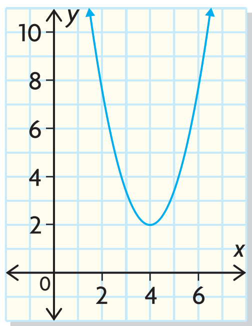
i)



iii)



ii)



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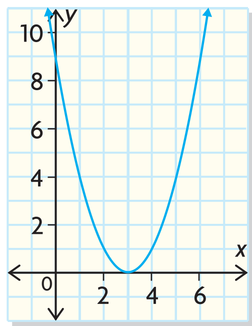
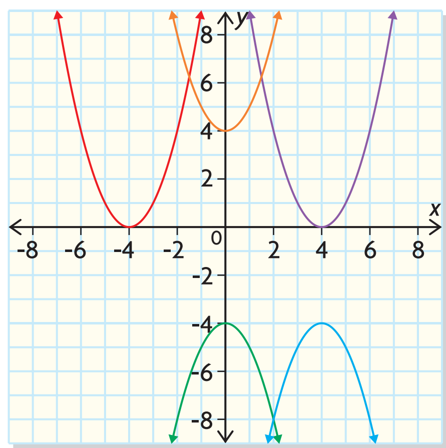


6. Explain how you would determine whether a parabola contains a minimum value or maximum value when the quadratic function that defines it is in vertex form:

$$y = a(x - h)^2 + k$$

Support your explanation with examples of functions and graphs.

7. State the equation of each function, if all the parabolas are congruent and if $a = 1$ or $a = -1$.



8. Marleen and Candice are both 6 ft tall, and they play on the same college volleyball team. In a game, Candice set up Marleen with an outside high ball for an attack hit. Using a video of the game, their coach determined that the height of the ball above the court, in feet, on its path from Candice to Marleen could be defined by the function

$$h(x) = -0.03(x - 9)^2 + 8$$

where x is the horizontal distance, measured in feet, from one edge of the court.

a) Determine the axis of symmetry of the parabola.

b) Marleen hit the ball at its highest point. How high above the court was the ball when she hit it?

c) How high was the ball when Candice set it, if she was 2 ft from the edge of the court?

d) State the range for the ball's path between Candice and Marleen. Justify your answer.

9. a) Write quadratic functions that define three different parabolas, all with their vertex at $(3, -1)$.

b) Predict how the graphs of the parabolas will be different from each other.

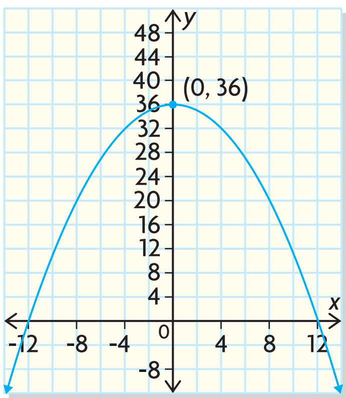
c) Graph each parabola on the same coordinate plane. How accurate were your predictions?

10. Without using a table of values or a graphing calculator, describe how you would graph the following function:

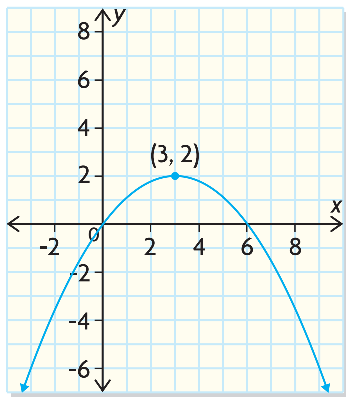
$$f(x) = 2(x - 1)^2 - 9$$

11. For each graph, determine the equation of the quadratic function in vertex form.

a)



b)



12. The vertex of a parabola is at $(4, -12)$.
a) Write a function to define all the parabolas with this vertex.
b) A parabola with this vertex passes through point $(13, 15)$. Determine the function for the parabola.
c) State the domain and range of the function you determined in part b).
d) Graph the quadratic function you determined in part b).

13. The height of the water, $h(t)$, in metres, that is sprayed from a sprinkler at a local golf course, can be modelled by the function

$$h(t) = -4.9(t - 1.5)^2 + 11.3$$

where time, t, is measured in seconds.
a) Graph the function, and estimate the zeros of the function.
b) What do the zeros represent in this situation?

14. A parabolic arch has x-intercepts $x = -6$ and $x = -1$. The parabola has a maximum height of 15 m.
a) Determine the quadratic function that models the parabola.
b) State the domain and range of the function.

15. Serge and a friend are throwing a paper airplane to each other. They stand 5 m apart from each other and catch the airplane at a height of 1 m above the ground. Serge throws the airplane on a parabolic flight path that achieves a minimum height of 0.5 m halfway to his friend.
a) Determine a quadratic function that models the flight path for the height of the airplane.
b) Determine the height of the plane when it is a horizontal distance of 1 m from Serge's friend.
c) State the domain and range of the function.

Closing

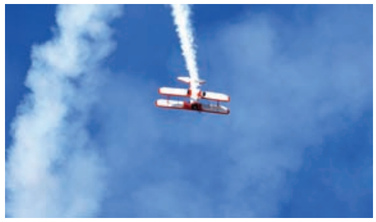
16. Liz claims that she can sketch an accurate graph more easily if a quadratic function is given in vertex form, rather than in standard or factored form. Do you agree or disagree? Explain.

An atlatl is used to launch a dart. It has been used as a hunting tool by peoples all over the world for thousands of years.

Extending

17. Peter is studying the flight path of an atlatl dart for a physics project. In a trial toss on the sports field, Peter threw his dart 80 yd and hit a platform that was 2 yd above the ground. The maximum vertical height of the atlatl was 10 yd. The dart was 2 yd above the ground when released.
a) Sketch a graph that models the flight path of the dart thrown by Peter.
b) How far from Peter, horizontally, was the atlatl dart when it reached a vertical height of 8 yd? Explain.

18. When an airplane is accelerated downward by combining its engine power with gravity, the airplane is said to be in a power dive. At the Abbotsford International Air Show, one of the stunt planes began such a manoeuvre. Selected data from the plane's flight log is shown below.



t	0	4	8	16
$h(t)$	520	200	40	200

a) Define a function, $h(t)$, that models the height of the plane above the ground, in metres, over time, t, in seconds, after the manoeuvre began.

b) How low to the ground did the plane get on this manoeuvre?

c) How long did it take for the plane to return to its initial altitude?

19. A bridge is going to be constructed over a river, as shown below. The supporting arch of the bridge will form a parabola. At the point where the bridge is going to be constructed, the river is 20 m wide from bank to bank. The arch will be anchored on the ground, 4 m from the edge of the riverbank on each side. The maximum height of the arch can be between 18 m and 22 m above the surface of the water. Create two different quadratic functions that model the supporting arch. Include a labelled graph for each arch.