7.3 Solving Quadratic Equations by Graphing

GOAL

Solve quadratic equations by graphing the corresponding function.

YOU WILL NEED

- graphing technology
- graph paper

EXPLORE...

- Graph the quadratic function $y = x^2 + 5$. How could you use your graph to solve the equation $21 = x^2 + 5$? What are some other equations you could solve with your graph?

INVESTIGATE the Math

Bonnie launches a model rocket from the ground with an initial velocity of 68 m/s. The following function, $h(t)$, can be used to model the height of the rocket, in metres, over time, t, in seconds:

$$h(t) = -4.9t^2 + 68t$$

Bonnie's friend Sasha is watching from a lookout point at a safe distance. Sasha's eye level is 72 m above the ground.

? How can you determine the times during the flight when the rocket will be at Sasha's eye level?

A. What is the value of $h(t)$ when the rocket is at Sasha's eye level?

B. Substitute the value of $h(t)$ that you calculated in part A into the function

$$h(t) = -4.9t^2 + 68t$$

to create a **quadratic equation**. You can solve this quadratic equation to determine when the rocket is at Sasha's eye level. Rewrite the quadratic equation in standard form.

C. Graph the function that corresponds to your equation. Use the **zeros** of the function to determine the t-intercepts.

D. Graph $h(t) = -4.9t^2 + 68t$. On the same axes, graph the horizontal line that represents Sasha's eye level. Determine the t-coordinates of the points where the two graphs intersect.

E. What do you notice about the t-coordinates of these points?

F. When will the rocket be at Sasha's eye level?

quadratic equation

A polynomial equation of the second degree; the standard form of a quadratic equation is
$ax^2 + bx + c = 0$
For example:
$2x^2 + 4x - 3 = 0$

zero

In a function, a value of the variable that makes the value of the function equal to zero.

Reflecting

G. How were your two graphs similar? How were they different?

H. Describe the two different strategies you used to solve the problem. What are the advantages of each?

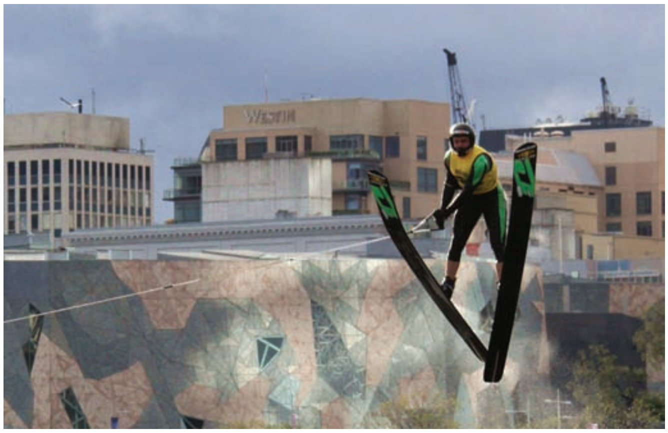
APPLY the Math

EXAMPLE 1 Verifying solutions to a quadratic equation

The flight time for a long-distance water ski jumper depends on the initial velocity of the jump and the angle of the ramp. For one particular jump, the ramp has a vertical height of 5 m above water level. The height of the ski jumper in flight, $h(t)$, in metres, over time, t, in seconds, can be modelled by the following function:

$$h(t) = 5.0 + 24.46t - 4.9t^2$$

How long does this water ski jumper hold his flight pose?



The skier holds his flight pose until he is 4.0 m above the water.

Olana's Solution

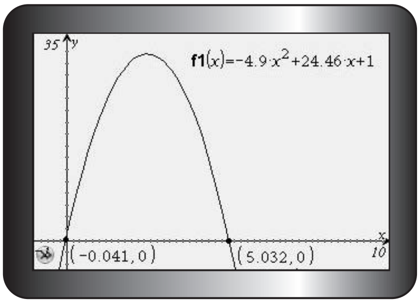
$$h(t) = 5.0 + 24.46t - 4.9t^2$$
$$4.0 = 5.0 + 24.46t - 4.9t^2$$

I substituted 4.0 for $h(t)$ to get a quadratic equation I can use to determine the time when the skier's height above the water is 4.0 m.

$$0 = 1.0 + 24.46t - 4.9t^2$$

I subtracted 4.0 from both sides to put the equation in standard form.

In standard form, $h(t) = 0$. Therefore, the solutions to the equation are the t-intercepts of the graph of this function.



I graphed the function on a calculator. I adjusted the window to show the vertex and the *x*-intercepts. I used the calculator to determine the *x*-intercepts.

The t-intercepts are 5.032 and -0.041.

I reread the problem to make sure each solution made sense. Time can't be negative in this situation, so the jumper did not come out of his pose at -0.041 s. Although $(-0.041, 0)$ is a point on the graph, it doesn't make sense in the context of this problem.

Verify:

$4.0 = 5.0 + 24.46t - 4.9t^2$

$t = 5.032$

LS	RS
4.0	$5.0 + 24.46(5.032) - 4.9(5.032)^2$
	$5.0 + 123.082 \ldots - 124.073 \ldots$
	$4.009 \ldots$

LS $\doteq$ RS

I verified the other solution by substituting it into the original equation. The left side was not quite equal to the right side, but I knew that this was because the calculator is set to show values to three decimal places. The solution is not exact, but it is correct.

The ski jumper holds his flight pose for about 5 s.

Your Turn

Curtis rearranged the equation $4.0 = 5.0 + 24.46t - 4.9t^2$ a different way and got the following equation:

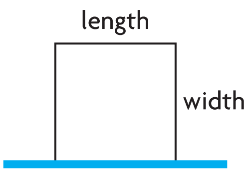
$$4.9t^2 - 24.46t - 1.0 = 0$$

a) Graph the function that is represented by Curtis's equation. How does this graph compare with Olana's graph?

b) Will Curtis get the same solution that Olana did? Explain.

EXAMPLE 2 Graphing to determine the number of roots

Lamont runs a boarding kennel for dogs. He wants to construct a rectangular play space for the dogs, using 40 m of fencing and an existing fence as one side of the play space.



a) Write a function that describes the area, A, in square metres, of the play space for any width, w, in metres.

b) Write equations you could use to determine the widths for areas of 250 m^2, 200 m^2, and 150 m^2.

c) Determine the number of possible widths for each equation using a graph.

Lamont's Solution

Let A represent the area of the play space in square metres.
Let l and w represent the dimensions of the play space in metres.

a)

$$l + 2w = 40$$
$$l = 40 - 2w$$

From the diagram, I could see that the total length of fencing can be expressed as two widths plus one length. I needed a function that just used variables for area and width, so I rewrote my equation to isolate l.

$$lw = A$$
$$(40 - 2w)w = A$$
$$40w - 2w^2 = A$$

I wrote the formula for the area of the play space and substituted $40 - 2w$ for l. Then I simplified the equation.

b)

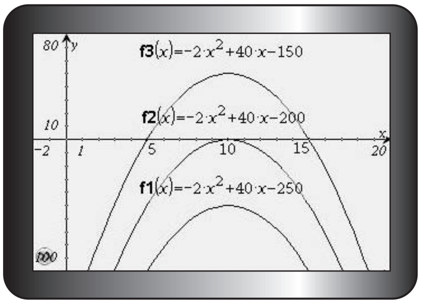
$$40w - 2w^2 = 250$$
$$-2w^2 + 40w - 250 = 0$$

$$40w - 2w^2 = 200$$
$$-2w^2 + 40w - 200 = 0$$

$$40w - 2w^2 = 150$$
$$-2w^2 + 40w - 150 = 0$$

To determine the equation for each area, I substituted the area for A. Then I rewrote each quadratic equation in standard form.

c)



I graphed the corresponding function for each equation.

I can't make a play space with an area of 250 m^2 using 40 m of fencing.

The graph of the first function,

$f_1(w) = -2w^2 + 40w - 250$,

did not cross the w-axis. There are no w-intercepts, so there are no solutions, or **roots**, to the equation.

If I make the play space 10 m wide, the area will be 200 m^2.

The graph of the next function,

$f_2(w) = -2w^2 + 40w - 200$,

intersected the w-axis at its vertex. There is one w-intercept, $w = 10$, so there is one root.

If I make the play space 5 m wide or 15 m wide, the area will be 150 m^2.

The graph of the third function,

$f_3(w) = -2w^2 + 40w - 150$,

has two w-intercepts, $w = 5$ and $w = 15$. This equation has two roots.

roots

The values of the variable that make an equation in standard form equal to zero. These are also called solutions to the equation. These values are also the zeros of the corresponding function and the x-intercepts of its graph.

Your Turn

Is it possible for a quadratic equation to have more than two roots? Use a graph to explain.

EXAMPLE 3 Solving a quadratic equation in non-standard form

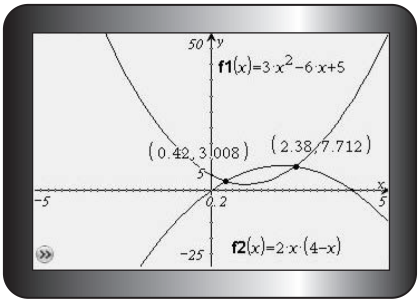
Determine the roots of this quadratic equation. Verify your answers.

$$3x^2 - 6x + 5 = 2x(4 - x)$$

Marwa's Solution

$f(x) = 3x^2 - 6x + 5$
$g(x) = 2x(4 - x)$

I wrote corresponding functions, $f(x)$ and $g(x)$, for each side of the equation to determine the roots.



I graphed each function on a calculator. Then I used the calculator to determine the points of intersection.

The solutions are $x = 0.420$ and $x = 2.380$.

I knew that the solutions of the quadratic equation are the x-coordinates of the points of intersection.

Verify:
$3x^2 - 6x + 5 = 2x(4 - x)$
$x = 0.420$

LS	RS
$3(0.420)^2 - 6(0.420) + 5$	$2(0.420)(4 - 0.420)$
3.009 ...	3.007 ...

LS $\doteq$ RS

Verify:
$3x^2 - 6x + 5 = 2x(4 - x)$
$x = 2.380$

LS	RS
$3(2.380)^2 - 6(2.380) + 5$	$2(2.380)(4 - 2.380)$
7.713 ...	7.711 ...

LS $\doteq$ RS

I verified the roots by substituting them into the original equation. Both solutions are valid.

The roots are $x = 0.420$ and $x = 2.380$.

Your Turn

Rewrite $3x^2 - 6x + 5 = 2x(4 - x)$ in standard form. If you graphed the function that corresponds to your equation in standard form, what x-intercepts would you expect to see? Why?

In Summary

Key Ideas

- A quadratic equation can be solved by graphing the corresponding quadratic function.
- The standard form of a quadratic equation is

$$ax^2 + bx + c = 0$$

- The roots of a quadratic equation are the x-intercepts of the graph of the corresponding quadratic function. They are also the zeros of the corresponding quadratic function.

Need to Know

- The zeros of a quadratic function correspond to the x-intercepts of the parabola that is defined by the function.
- If a quadratic equation is in standard form
  - you can graph the corresponding quadratic function and determine the zeros of the function to solve the equation
- If the quadratic function is not in standard form
  - you can graph the expression on the left side and the expression on the right side as functions on the same axes
  - the x-coordinates of the points of intersection of the two graphs are the roots of the equation
- For any quadratic equation, there can be zero, one, or two real roots. This is because a parabola can intersect the x-axis in zero, one, or two places.

CHECK Your Understanding

1. Solve each equation by graphing the corresponding function and determining the zeros.

a) $2x^2 - 5x - 3 = 0$ **b)** $9x - 4x^2 = 0$

2. Solve each equation by graphing the expressions on both sides of the equation.

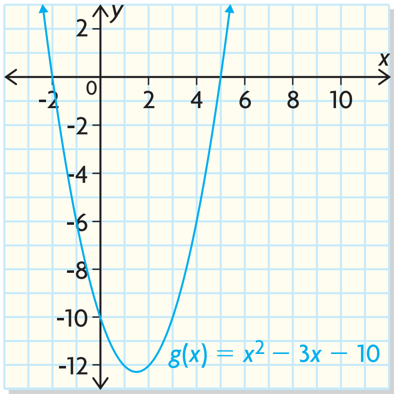
a) $x^2 + 5x = 24$ **b)** $0.5x^2 = -2x + 3$

3. Rewrite each equation in standard form. Then solve the equation in standard form by graphing.

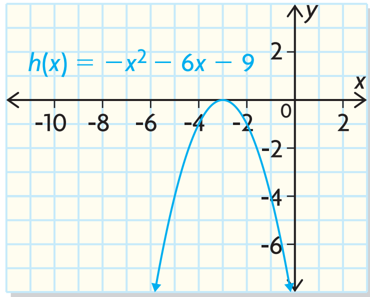
a) $6a^2 = 11a + 35$ **b)** $2p^2 + 3p = 1 - 2p$

4. For each graph, determine the roots of the corresponding quadratic equation.

a)



b)



PRACTISING

5. Solve each equation by graphing the corresponding function and determining the zeros.

a) $3x^2 - 6x - 7 = 0$

b) $0.5z^2 + 3z - 2 = 0$

c) $3b^2 + 8b + 7 = 0$

d) $0.09x^2 + 0.30x + 0.25 = 0$

6. Solve each equation by graphing the expressions on both sides of the equation.

a) $3a^2 = 18a - 21$

b) $5p = 3 - 2p^2$

c) $4x(x + 3) = 3(4x + 3)$

d) $x^2 - 3x - 8 = -2x^2 + 8x + 1$

7. A ball is thrown into the air from a bridge that is 14 m above a river. The function that models the height, $h(t)$, in metres, of the ball over time, t, in seconds is

$$h(t) = -4.9t^2 + 8t + 14$$

a) When is the ball 16 m above the water?

b) When is the ball 12 m above the water? Explain.

c) Is the ball ever 18 m above the water? Explain how you know.

d) When does the ball hit the water?

8. Solve each quadratic equation by graphing.

a) $5x^2 - 2x = 4x + 3$

b) $-2x^2 + x - 1 = x^2 - 3x - 7$

c) $3x^2 - 12x + 17 = -4(x - 2)^2 + 5$

d) $5x^2 + 4x + 3 = -x^2 - 2x$

9. The stopping distance, d, of a car, in metres, depends on the speed of the car, s, in kilometres per hour. For a certain car on a dry road, the equation for stopping distance is

$$d = 0.0059s^2 + 0.187s$$

The driver of the car slammed on his brakes to avoid an accident, creating skid marks that were 120 m long. He told the police that he was driving at the speed limit of 100 km/h. Do you think he was speeding? Explain.

10. Solve the following quadratic equation using the two methods described below.

$$4x^2 + 3x - 2 = -2x^2 + 5x + 1$$

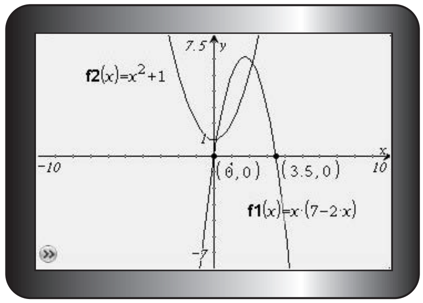
a) Graph the expressions on both sides of the equation, and determine the points of intersection.

b) Rewrite the quadratic equation in standard form, graph the corresponding function, and determine the zeros.

c) Which method do you prefer for this problem? Explain.

11. The length of a rectangular garden is 4 m more than its width. Determine the dimensions of the garden if the area is 117 m^2.

12. Kevin solved the following quadratic equation by graphing the expressions on both sides on the same axes.



$$x(7 - 2x) = x^2 + 1$$

His solutions were $x = 0$ and $x = 3.5$. When he verified his solutions, the left side did not equal the right side.

Verify:

$x(7 - 2x) = x^2 + 1$

$x = 0$

LS	RS
$x(7 - 2x)$	$x^2 + 1$
$(0)(7 - 2(0))$	$(0)^2 + 1$
$(0)(7)$	$0 + 1$
0	1

LS $\neq$ RS

$x = 3.5$

LS	RS
$x(7 - 2x)$	$x^2 + 1$
$(3.5)(7 - 2(3.5))$	$(3.5)^2 + 1$
$(3.5)(7 - 7)$	$12.25 + 1$
0	13.25

LS $\neq$ RS

a) Identify Kevin's error.

b) Determine the correct solution.

13. Solve each equation.

a) $0.25x^2 - 1.48x - 178 = 0$

b) $4.9x(6 - x) + 36 = 2(x + 9) - x^2$

Closing

14. Explain how you could use a graph to determine the number of roots for an equation in the form $ax^2 + bx = c$.

Extending

15. On the same axes, graph these quadratic functions:

$$y = -2x^2 + 20x - 42$$
$$y = x^2 - 10x + 21$$

Write three different equations whose roots are the points of intersection of these graphs.