7.2 Properties of Graphs of Quadratic Functions

GOAL

Identify the characteristics of graphs of quadratic functions, and use the graphs to solve problems.

YOU WILL NEED

- ruler
- graph paper

LEARN ABOUT the Math

Nicolina plays on her school's volleyball team. At a recent match, her Nonno, Marko, took some time-lapse photographs while she warmed up. He set his camera to take pictures every 0.25 s. He started his camera at the moment the ball left her arms during a bump and stopped the camera at the moment that the ball hit the floor. Marko wanted to capture a photo of the ball at its greatest height. However, after looking at the photographs, he could not be sure that he had done so. He decided to place the information from his photographs in a table of values.

Time (s)	Height (ft)
0.00	2
0.25	6
0.50	8
0.75	8
1.00	6
1.25	2

From his photographs, Marko observed that Nicolina struck the ball at a height of 2 ft above the ground. He also observed that it took about 1.25 s for the ball to reach the same height on the way down.

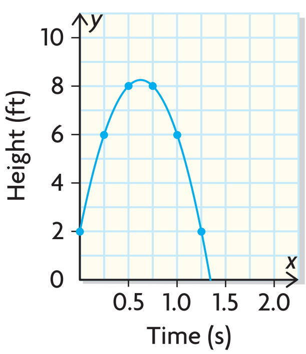
? When did the volleyball reach its greatest height?

EXPLORE...

- Parabolic skis are marketed as performing better than traditional straight-edge skis. Parabolic skis are narrower in the middle than on the ends. Design one side of a parabolic ski on a coordinate grid. In groups, discuss where any lines of symmetry occur and how the parabolic shape works in your design.

EXAMPLE 1 Using symmetry to estimate the coordinates of the vertex

Marko's Solution

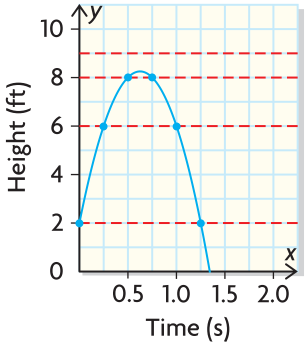


I plotted the points from my table, and then I sketched a graph that passed through all the points.

The graph looked like a parabola, so I concluded that the relation is probably quadratic.

vertex

The point at which the quadratic function reaches its maximum or minimum value.



I knew that I could draw horizontal lines that would intersect the parabola at two points, except at the **vertex**, where a horizontal line would intersect the parabola at only one point.

Using a ruler, I drew horizontal lines and estimated that the coordinates of the vertex are around (0.6, 8.2).

This means that the ball reached maximum height at just over 8 ft, about 0.6 s after it was launched.

Equation of the axis of symmetry:

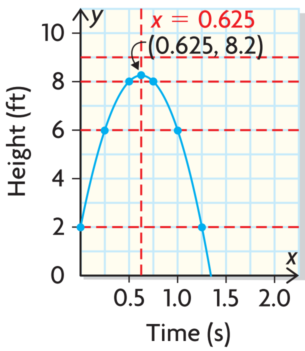
$$x = \frac{0 + 1.25}{2}$$

$$x = 0.625$$

I used points that have the same y-value, (0, 2) and (1.25, 2), to determine the equation of the **axis of symmetry**. I knew that the axis of symmetry must be the same distance from each of these points.

axis of symmetry

A line that separates a 2-D figure into two identical parts. For example, a parabola has a vertical axis of symmetry passing through its vertex.



From the equation, the x-coordinate of the vertex is 0.625. From the graph, the y-coordinate of the vertex is close to 8.2.

Therefore, 0.625 s after the volleyball was struck, it reached its maximum height of approximately 8 ft 2 in.

I revised my estimate of the coordinates of the vertex.

Reflecting

A. How could Marko conclude that the graph was a quadratic function?

B. If a horizontal line intersects a parabola at two points, can one of the points be the vertex? Explain.

C. Explain how Marko was able to use symmetry to determine the time at which the volleyball reached its maximum height.

APPLY the Math

EXAMPLE 2 Reasoning about the maximum value of a quadratic function

Some children are playing at the local splash pad. The water jets spray water from ground level. The path of water from one of these jets forms an arch that can be defined by the function

$$f(x) = -0.12x^2 + 3x$$

where x represents the horizontal distance from the opening in the ground in feet and $f(x)$ is the height of the sprayed water, also measured in feet. What is the maximum height of the arch of water, and how far from the opening in the ground can the water reach?

Manuel's Solution

$$f(x) = -0.12x^2 + 3x$$

I knew that the degree of the function is 2, so the function is quadratic. The arch must be a parabola.

I also knew that the coefficient of x^2, a, is negative, so the parabola opens down. This means that the function has a **maximum value**, associated with the y-coordinate of the vertex.

$$f(0) = 0$$

$$f(1) = -0.12(1)^2 + 3(1)$$
$$f(1) = -0.12 + 3$$
$$f(1) = 2.88$$

x	0	1	2	12	13
$f(x)$	0	2.88	5.52	18.72	18.72

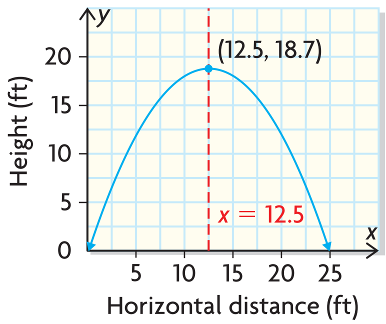
I started to create a table of values by determining the y-intercept. I knew that the constant, zero, is the y-coordinate of the y-intercept. This confirms that the stream of water shoots from ground level.

I continued to increase x by intervals of 1 until I noticed a repeat in my values. A height of 18.72 ft occurs at horizontal distances of 12 ft and 13 ft.

Based on symmetry and the table of values, the maximum value of $f(x)$ will occur halfway between (12, 18.72) and (13, 18.72).

The arch of water will reach a maximum height between 12 ft and 13 ft from the opening in the ground.

maximum value
The greatest value of the dependent variable in a relation.



I used my table of values to sketch the graph. I extended the graph to the x-axis. I knew that my sketch represented only part of the function, since I am only looking at the water when it is above the ground.

$$x = \frac{12 + 13}{2}$$
$$x = 12.5$$

I used two points with the same y-value, (12, 18.72) and (13, 18.72), to determine the equation of the axis of symmetry.

Equation of the axis of symmetry:
$x = 12.5$

Height at the vertex:

$$f(x) = -0.12x^2 + 3x$$
$$f(12.5) = -0.12(12.5)^2 + 3(12.5)$$
$$f(12.5) = -0.12(156.25) + 37.5$$
$$f(12.5) = -18.75 + 37.5$$
$$f(12.5) = 18.75$$

I knew that the x-coordinate of the vertex is 12.5, so I substituted 12.5 into the equation to determine the height of the water at this horizontal distance.

The water reaches a maximum height of 18.75 ft when it is 12.5 ft from the opening in the ground.

The water can reach a maximum horizontal distance of 25 ft from the opening in the ground.

Due to symmetry, the opening in the ground must be the same horizontal distance from the axis of symmetry as the point on the ground where the water lands. I simply multiplied the horizontal distance to the axis of symmetry by 2.

The domain of this function is $0 \leq x \leq 25$, where $x \in \mathbb{R}$.

Your Turn

Another water arch at the splash pad is defined by the following quadratic function:

$$f(x) = -0.15x^2 + 3x$$

a) Graph the function, and state its domain for this context.
b) State the range for this context.
c) Explain why the original function describes the path of the water being sprayed, whereas the function in *Example 1* does not describe the path of the volleyball.

EXAMPLE 3 Graphing a quadratic function using a table of values

Sketch the graph of the function:

$$y = x^2 + x - 2$$

Determine the y-intercept, any x-intercepts, the equation of the axis of symmetry, the coordinates of the vertex, and the domain and range of the function.

Anthony's Solution

$y = x^2 + x - 2$

The function is a quadratic function in the form

$y = ax^2 + bx + c$
$a = 1$
$b = 1$
$c = -2$

The degree of the given equation is 2, so the graph will be a parabola.

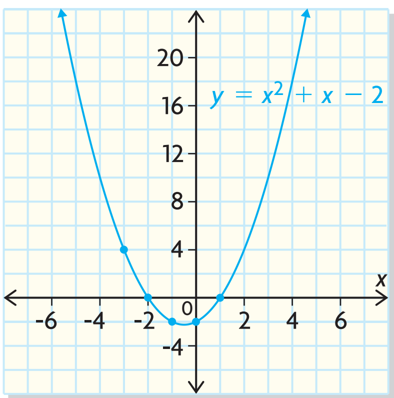
Since the coefficient of x^2 is positive, the parabola opens up.

Since the y-coordinate of the y-intercept is less than zero and the parabola opens up, there must be two x-intercepts and a **minimum value**.

x	−3	−2	−1	0	1
y	4	0	−2	−2	0

I made a table of values. I included the y-intercept, (0, −2), and determined some other points by substituting values of x into the equation.

I stopped determining points after I had identified both x-intercepts, because I knew that I had enough information to sketch an accurate graph.



I graphed each coordinate pair and then drew a parabola that passed through all the points.

Equation of the axis of symmetry:

$$x = \frac{-2 + 1}{2}$$

$$x = \frac{-1}{2}$$

$$x = -0.5$$

I used the x-intercepts to determine the equation of the axis of symmetry.

minimum value

The least value of the dependent variable in a relation.

y-coordinate of the vertex:

$y = (-0.5)^2 + (-0.5) - 2$
$y = 0.25 - 0.5 - 2$
$y = -2.25$
The vertex is $(-0.5, -2.25)$.

I knew that the vertex is a point on the axis of symmetry. The x-coordinate of the vertex must be -0.5. To determine the y-coordinate of the vertex, I substituted -0.5 for x in the given equation.

The y-intercept is -2.
The x-intercepts are -2 and 1.
The equation of the axis of symmetry is
$x = -0.5$
The vertex is $(-0.5, -2.25)$.

The vertex, $(-0.5, -2.25)$, defines the minimum value of y.

Domain and range:
$\{(x, y) \mid x \in R, y \geq -2.25, y \in R\}$

No restrictions were given for x, so the domain is all real numbers.

Your Turn

Explain how you could decide if the graph of the function $y = -x^2 + x + 2$ has x-intercepts.

EXAMPLE 4 Locating a vertex using technology

A skier's jump was recorded in frame-by-frame analysis and placed in one picture, as shown.

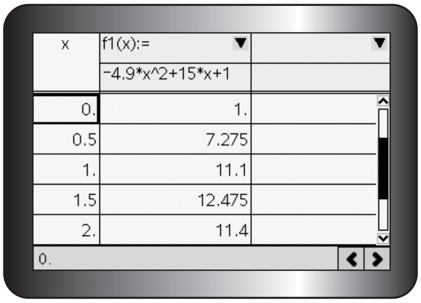
The skier's coach used the picture to determine the quadratic function that relates the skier's height above the ground, y, measured in metres, to the time, x, in seconds that the skier was in the air:

$$y = -4.9x^2 + 15x + 1$$

Graph the function. Then determine the skier's maximum height, to the nearest tenth of a metre, and state the range of the function for this context.

Isidro's Solution

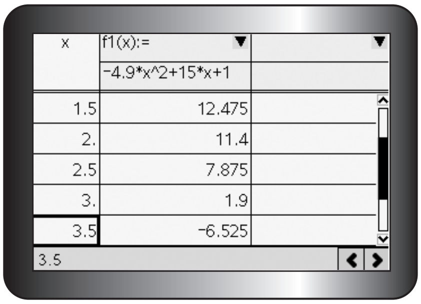
$y = -4.9x^2 + 15x + 1$

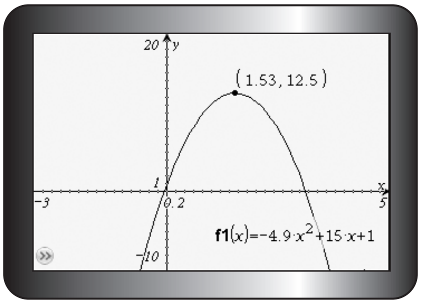


I entered the equation into my calculator.

To make sure that the graph models the situation, I set up a table of values. The skier's jump will start being timed at 0 s, and the skier will be in the air for only a few seconds, so I set the table to start at an *x*-value of zero and to increase in increments of 0.5.

I decided to set the minimum height at 0 m—it doesn't make sense to extend the function below the *x*-axis, because the skier cannot go below the ground. I checked the table and noticed that the greatest *y*-value is only 12.475... m, and that *y* is negative at 3.5 s. I used these values to set an appropriate viewing window for the graph.





I graphed the function and used the calculator to locate the maximum value of the function.

The skier achieved a maximum height of 12.5 m above the ground 1.5 s into the jump.

The range of the function is $\{y \mid 0 \leq y \leq 12.5, y \in \mathrm{R}\}$.

In this situation, the height of the skier varies between 0 m and 12.5 m.

Your Turn

On the next day of training, the coach asked the skier to increase his speed before taking the same jump. At the end of the day, the coach analyzed the results and determined the equation that models the skier's best jump:

$$y = -4.9x^2 + 20x + 1$$

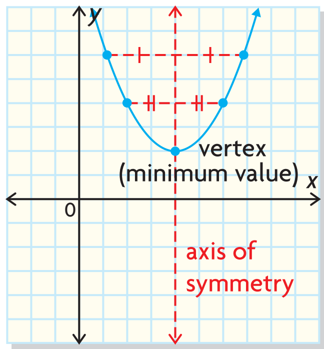
How much higher did the skier go on this jump?

In Summary

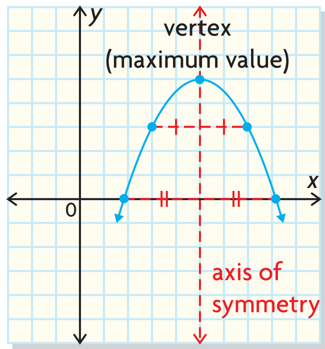
Key Idea

- A parabola that is defined by the equation $y = ax^2 + bx + c$ has the following characteristics:
  - If the parabola opens down ($a < 0$), the vertex of the parabola is the point with the greatest y-coordinate. The y-coordinate of the vertex is the maximum value of the function.
  - If the parabola opens up ($a > 0$), the vertex of the parabola is the point with the least y-coordinate. The y-coordinate of the vertex is the minimum value of the function.
  - The parabola is symmetrical about a vertical line, the axis of symmetry, through its vertex.

$y = ax^2 + bx + c$
$a > 0$



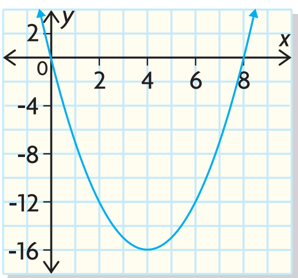
$y = ax^2 + bx + c,$
$a < 0$



Need to Know

- For all quadratic functions, the domain is the set of real numbers, and the range is a subset of real numbers.
- When a problem can be modelled by a quadratic function, the domain and range of the function may need to be restricted to values that have meaning in the context of the problem.

CHECK Your Understanding

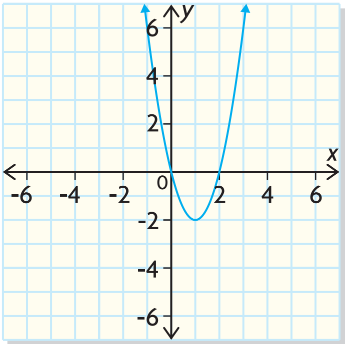


1. a) Determine the equation of the axis of symmetry for the parabola.
   b) Determine the coordinates of the vertex of the parabola.
   c) State the domain and range of the function.

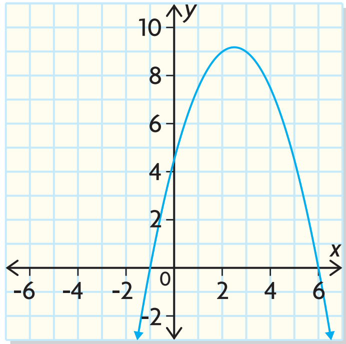
2. State the coordinates of the y-intercept and two additional ordered pairs for each function.
   a) $f(x) = 2x^2 + 8x + 8$ **b)** $f(x) = 4x - x^2$

3. For each function, identify the x- and y-intercepts, determine the equation of the axis of symmetry and the coordinates of the vertex, and state the domain and range.

a)



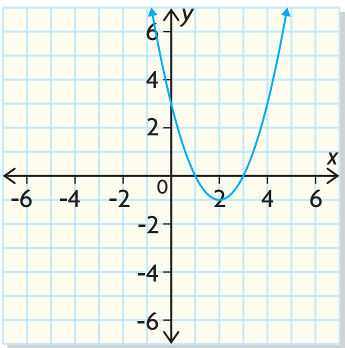
b)



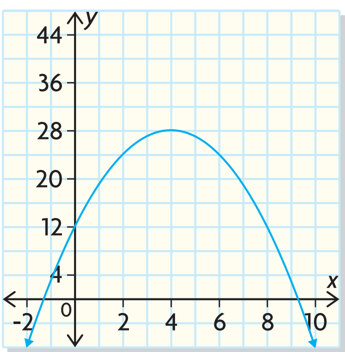
PRACTISING

4. For each function, identify the equation of the axis of symmetry, determine the coordinates of the vertex, and state the domain and range.

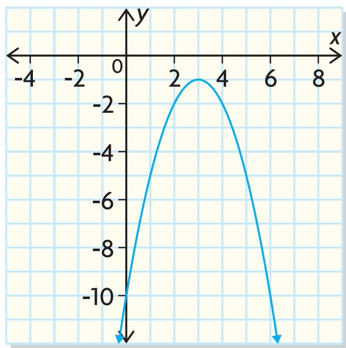
a)



b)



c)



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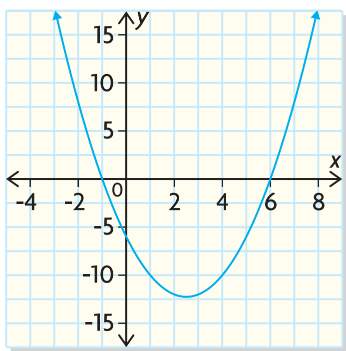
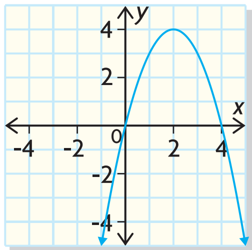


5. Each parabola in question 4 is defined by one of the functions below.

a) $f(x) = x^2 - 5x - 6$
b) $f(x) = -x^2 + 8x + 12$
c) $f(x) = -x^2 + 6x - 10$
d) $f(x) = x^2 - 4x + 3$

Identify the function that defines each graph. Then verify the coordinates of the vertex that you determined in question 4.

6. State whether each parabola has a minimum or maximum value, and then determine this value.

a)


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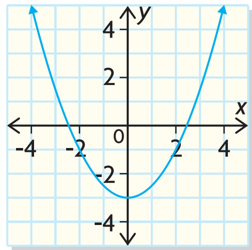

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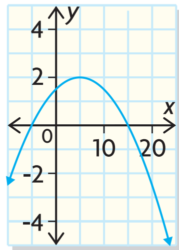

x	−4	−2	0	2	4
y					

7. a) Complete the table of values shown for each of the following functions.

i) $y = -\frac{1}{2}x^2 + 5$ **ii)** $y = \frac{3}{2}x^2 - 2$

b) Graph the points in your table of values.
c) State the domain and range of the function.

8. a) Graph the functions $y = 2x^2$ and $y = -2x^2$.
b) How are the graphs the same? How are the graphs different?
c) Suppose that the graphs were modified so that they became the graphs of $y = 2x^2 + 4$ and $y = -2x^2 + 4$. Predict the vertex of each function, and explain your prediction.

9. For each of the following, both points, (x, y), are located on the same parabola. Determine the equation of the axis of symmetry for each parabola.

a) (0, 2) and (6, 2)
b) (1, −3) and (9, −3)
c) (−6, 0) and (2, 0)
d) (−5, −1) and (3, −1)

10. A parabola has *x*-intercepts $x = 3$ and $x = -9$. Determine the equation of the axis of symmetry for the parabola.

11. a) Graph each function.

i) $f(x) = 2x^2 + 3$
ii) $f(x) = -x^2 - 7x + 4$
iii) $f(x) = x^2 - 6x + 9$
iv) $f(x) = \frac{1}{2}x^2 - 4x + 3$

b) Determine the equation of the axis of symmetry and the coordinates of the vertex for each parabola.
c) State the domain and range of each function.

12. In southern Alberta, near Fort Macleod, you will find the famous Head-Smashed-In Buffalo Jump. In a form of hunting, Blackfoot once herded buffalo and then stampeded the buffalo over the cliffs. If the height of a buffalo above the base of the cliff, $f(x)$, in metres, can be modelled by the function

$$f(x) = -4.9x^2 + 12$$

where x is the time in seconds after the buffalo jumped, how long was the buffalo in the air, to the nearest hundredth of a second?

13. In the game of football, a team can score by kicking the ball over a bar and between two uprights. For a kick in a particular game, the height of the ball above the ground, y, in metres, can be modelled by the function

$$y = -4.9x^2 + 25x$$

where x is the time in seconds after the ball left the foot of the player.

a) Determine the maximum height that this kick reached, to the nearest tenth of a metre.

b) State any restrictions that the context imposes on the domain and range of the function.

c) How long was the ball in the air?

14. An annual fireworks festival, held near the seawall in downtown Vancouver, choreographs rocket launches to music. The height of one rocket, $h(t)$, in metres over time, t, in seconds, is modelled by the function

$$h(t) = -4.9t^2 + 80t$$

Determine the domain and range of the function that defines the height of this rocket, to the nearest tenth of a metre.

15. Melinda and Genevieve live in houses that are next to each other. Melinda lives in a two-storey house, and Genevieve lives in a bungalow. They like to throw a tennis ball to each other through their open windows. The height of a tennis ball thrown from Melinda to Genevieve, $f(x)$, in feet, over time, x, measured in seconds is modelled by the function

$$f(x) = -5x^2 + 6x + 12$$

What are the domain and range of this function if Genevieve catches the ball 4 ft above the ground? Draw a diagram to support your answer.

16. Sid knows that the points $(-1, 41)$ and $(5, 41)$ lie on a parabola defined by the function

$$f(x) = 4x^2 - 16x + 21$$

a) Does $f(x)$ have a maximum value or a minimum value? Explain.
b) Determine, in two different ways, the coordinates of the vertex of the parabola.

Closing

17. a) Explain the relationship that must exist between two points on a parabola if the x-coordinates of the points can be used to determine the equation of the axis of symmetry for the parabola.
b) How can the equation of the axis of symmetry be used to determine the coordinates of the vertex of the parabola?

Extending

18. Gamez Inc. makes handheld video game players. Last year, accountants modelled the company's profit using the equation

$$P = -5x^2 + 60x - 135$$

This year, accountants used the equation

$$P = -7x^2 + 70x - 63$$

In both equations, P represents the profit, in hundreds of thousands of dollars, and x represents the number of game players sold, in hundreds of thousands. If the same number of game players were sold in these years, did Gamez Inc.'s profit increase? Justify your answer.

19. A parabola has a range of $\{y \mid y \le 14.5, y \in \mathbb{R}\}$ and a y-intercept of 10. The axis of symmetry of the parabola includes point $(-3, 5)$. Write the equation that defines the parabola in standard form if $a = \frac{-1}{2}$.