6.4 Optimization Problems I: Creating the Model

YOU WILL NEED

- graphing technology OR graph paper, ruler, and coloured pencils

EXPLORE...

- A florist is ordering bracken fern and baby's breath for bouquets and centrepieces.
- No more than 100 stems of baby's breath will be ordered.
- More than 100 stems of bracken fern will be ordered.
- The florist has space to store no more than 250 stems, in total.
- Is each of the following a combination she can order? Explain.

Baby's Breath	Bracken Fern
0	150
25	25
50	150
100	100
95.5	114.5
100	150
150	125

GOAL

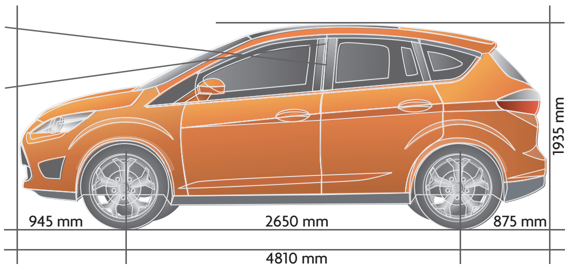
Create models to represent optimization problems.

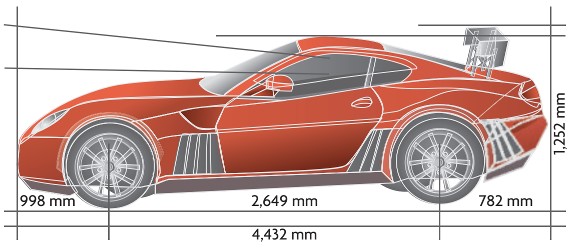
INVESTIGATE the Math

A toy company manufactures two types of toy vehicles: racing cars and sport-utility vehicles.

- Because the supply of materials is limited, no more than 40 racing cars and 60 sport-utility vehicles can be made each day.
- However, the company can make 70 or more vehicles, in total, each day.
- It costs $8 to make a racing car and $12 to make a sport-utility vehicle.

There are many possible combinations of racing cars and sport-utility vehicles that could be made. The company wants to know what combinations will result in the minimum and maximum costs, and what those costs will be.





? How can this situation be modelled?

A. What are the two variables in this situation?

B. Write a system of three linear inequalities to represent these conditions:
- the total number of racing cars that can be made
- the total number of sport-utility vehicles that can be made
- the total number of vehicles that can be made

C. What do you know about the restrictions on the domain and range of the variables? Explain.

D. Graph the system. Choose at least two points in the solution region that are possible solutions to the system.

E. What quantity in this situation needs to be minimized and maximized? Write an equation to represent how the two variables relate to this quantity.

Reflecting

F. Each combination below is a possible solution to the system of linear inequalities:
i) 40 racing cars and 60 sport-utility vehicles
ii) 40 racing cars and 30 sport-utility vehicles
iii) 10 racing cars and 60 sport-utility vehicles
iv) 30 racing cars and 40 sport-utility vehicles

Use your equation from part E to calculate the manufacturing cost for each solution. What do you notice?

APPLY the Math

EXAMPLE 1 Creating a model for an optimization problem with whole-number variables

Three teams are travelling to a basketball tournament in cars and minivans.

- Each team has no more than 2 coaches and 14 athletes.
- Each car can take 4 team members, and each minivan can take 6 team members.
- No more than 4 minivans and 12 cars are available.

The school wants to know the combination of cars and minivans that will require the minimum and maximum number of vehicles. Create a model to represent this situation.

Juanita's Solution

Let m represent the number of minivans.
Let c represent the number of cars.

The two variables in the problem are the number of cars and the number of minivans. The values of these variables are whole numbers.

$m \in W$ and $c \in W$

optimization problem

A problem where a quantity must be maximized or minimized following a set of guidelines or conditions.

constraint

A limiting condition of the optimization problem being modelled, represented by a linear inequality.

objective function

In an optimization problem, the equation that represents the relationship between the two variables in the system of linear inequalities and the quantity to be optimized.

feasible region

The solution region for a system of linear inequalities that is modelling an optimization problem.

Constraints:
Number of cars available:
$c \leq 12$
Number of minivans available:
$m \leq 4$
Number of team members:
$4c + 6m \leq 48$

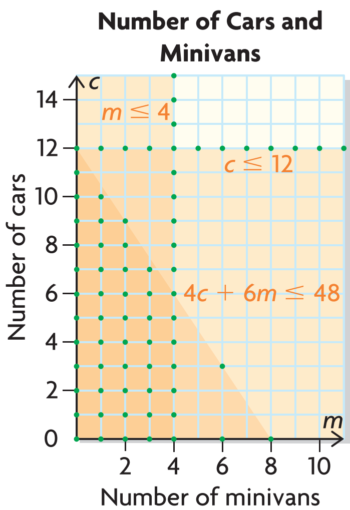
I knew that this is an **optimization problem** because the number of vehicles has to be minimized and maximized.

I wrote three linear inequalities to represent the three limiting conditions, or **constraints**.

The maximum number of team members is the number of teams multiplied by the maximum number of coaches and athletes:
$3(14) + 3(2) = 48$

Objective function:
Let V represent the total number of vehicles.
$V = c + m$

I created an equation, called the **objective function**, to represent the relationship between the two variables (number of minivans and number of cars) and the quantity to be minimized and maximized (number of vehicles).



I graphed the system of three inequalities.

One of the solutions in the **feasible region** represents the combination of cars and minivans that results in the minimum total number of vehicles and another solution represents the maximum. I think I could use the objective function to determine each point, but I am not certain how yet.

Your Turn

Suppose that the greatest number of athletes changed from 14 to 12 per team. How would Juanita's model change?

EXAMPLE 2 Creating a model for a maximization problem with positive real-number variables

A refinery produces oil and gas.

- At least 2 L of gasoline is produced for each litre of heating oil.
- The refinery can produce up to 9 million litres of heating oil and 6 million litres of gasoline each day.
- Gasoline is projected to sell for \$1.10 per litre. Heating oil is projected to sell for \$1.75 per litre.

The company needs to determine the daily combination of gas and heating oil that must be produced to maximize revenue. Create a model to represent this situation.

Umberto's Solution

Let h represent the number of litres of heating oil.
Let g represent the number of litres of gasoline.

Restrictions:
$h \geq 0$ and $g \geq 0$, where $h \in \mathrm{R}$ and $g \in \mathrm{R}$

I knew that this is an optimization problem because the total revenue has to be maximized.

The two variables in the problem are the volume of heating oil and the volume of gasoline, both in litres. Litres are measured using positive real numbers.

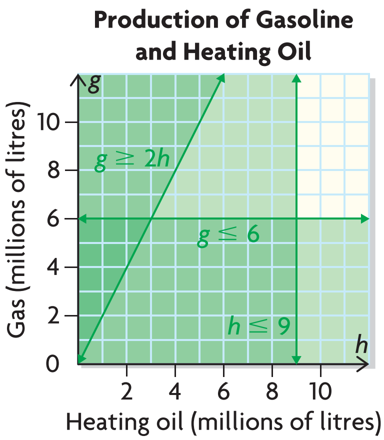
Constraints:
Ratio of gasoline produced to oil produced:
$g \geq 2h$
Amount of gasoline that can be produced:
$g \leq 6\ 000\ 000$
Amount of oil that can be produced:
$h \leq 9\ 000\ 000$

I created inequalities to represent the five constraints of the problem.

I treated the restrictions on each variable as a constraint.

Let R represent total revenue from sales of gasoline and heating oil.
Objective function to maximize:
$R = 1.10g + 1.75h$

I wrote an objective function to represent the relationship between the two variables (volume of heating oil and volume of gasoline) and the quantity to be maximized (total revenue).



I graphed the system of inequalities in the 1st quadrant because of the restrictions on the variables. The feasible region is a right triangle and includes all points on its boundaries.

I think I can use the objective function to determine which point in the feasible region represents the combination of oil and gas that will result in the maximum revenue, but I am not sure how yet.

Your Turn

Suppose that the refinery produced at least 2 L of heating oil for each litre of gasoline. Discuss with a partner how Umberto's model would change.

In Summary

Key Ideas

- To solve an optimization problem, you need to determine which combination of values of two variables results in a maximum or minimum value of a related quantity.
- When creating a model, the first step is to represent the situation algebraically. An algebraic model includes these parts:
  - a defining statement of the variables used in your model
  - a statement describing the restrictions on the variables
  - a system of linear inequalities that describes the constraints
  - an objective function that shows how the variables are related to the quantity to be optimized
- The second step is to represent the system of linear inequalities graphically.
- In optimization problems, any restrictions on the variables are considered constraints. For example, if you are working with positive real numbers, $x \geq 0$ and $y \geq 0$ are constraints and should be included in the system of linear inequalities.

Need to Know

- You can create a model for an optimization problem by following these steps:

  Step 1. Identify the quantity that must be optimized. Look for key words, such as *maximize* or *minimize*, *largest* or *smallest*, and *greatest* or *least*.

  Step 2. Define the variables that affect the quantity to be optimized. Identify any restrictions on these variables.

  Step 3. Write a system of linear inequalities to describe all the constraints of the problem. Graph the system.

  Step 4. Write an objective function to represent the relationship between the variables and the quantity to be optimized.

CHECK Your Understanding

1. Baskets of fruit are being prepared to sell.

- Each basket contains at least 5 apples and at least 6 oranges.
- Apples cost 20¢ each, and oranges cost 35¢ each. The budget allows no more than $7, in total, for the fruit in each basket.

Answer each part below to create a model that could be used to determine the combination of apples and oranges that will result in the maximum number of pieces of fruit in a basket.

a) What are the two variables in this situation? Describe any restrictions.

b) Write a system of linear inequalities to represent each constraint:

i) the number of apples in each basket

ii) the number of oranges in each basket

iii) the cost of each basket (in cents)

c) Graph the system.

d) Write the objective function that represents how the quantity to be maximized relates to the variables.

2. A fast-food concession stand sells hotdogs and hamburgers.

- Daily sales can be as high as 300 hamburgers and hot dogs combined.
- The stand has room to stock no more than 200 hot dogs and no more than 150 hamburgers.
- Hot dogs are sold for $3.25, and hamburgers are sold for $4.75.

Create a model that could be used to determine the combination of hamburgers and hot dogs that will result in maximum sales.

PRACTISING

3. A vending machine sells juice and pop.

- The machine holds, at most, 240 cans of drinks.
- Sales from the vending machine show that at least 2 cans of juice are sold for each can of pop.
- Each can of juice sells for $1.00, and each can of pop sells for $1.25.

Create a model that could be used to determine the maximum revenue from the vending machine.

4. A student council is ordering signs for the spring dance. Signs can be made in letter size or poster size.

- No more than 15 of each size are wanted.
- At least 15 signs are needed altogether.
- Letter-size signs cost $9.80 each, and poster-size signs cost $15.75 each.

Create a model that could be used to determine a combination of the two sizes of signs that would result in the lowest cost to the council.

5. A football stadium has 50 000 seats.
- Two-fifths of the seats are in the lower deck.
- Three-fifths of the seats are in the upper deck.
- At least 30 000 tickets are sold per game.
- A lower deck ticket costs $120, and an upper deck ticket costs $80.

Create a model that could be used to determine a combination of tickets for lower-deck and upper-deck seats that should be sold to maximize revenue.

6. Sung and Faith have weekend jobs at a marina, applying anti-fouling paint to the bottom of boats.
- Sung can work no more than 14 h per weekend.
- Faith is available no more than 18 h per weekend.
- The marina will hire both of them for 24 h or less per weekend.
- Sung paints one boat in 3 h, but Faith needs 4 h to paint one boat.

The marina wants to maximize the number of boats that are painted each weekend.

a) Create a model to represent this situation.

b) Suppose that another employee, Frank, who can paint a boat in 2 h, replaced Faith for a weekend. How would your model change?

7. A Saskatchewan farmer is planting wheat and barley.
- He wants to plant no more than 1000 ha altogether.
- The farmer wants at least three times as many hectares of wheat as barley.
- The yield per hectare of wheat averages 50 bushels, and the yield per hectare of barley averages 38 bushels.
- Wheat pays the farmer $5.25 per bushel, and barley pays $3.61 per bushel.

The farmer wants to plant a combination of wheat and barley that will maximize revenue. Create a model to represent this situation.

Closing

8. With a partner, develop a list of questions that could help you create a model for an optimization problem.

Extending

9. At an orienteering meet, two courses are run over two days: a long course on the first day and a shorter course on the second day. George wants to run the first course in no more than 16 min and the second course in no more than 12 min. He is aiming for a combined time of no more than 25 min. The winner has the lowest combined time and wins $25 for every minute under 30 min. Create a model to represent this situation.

Math in Action

Representing the Value of Beauty

Imagine a stream flowing slowly around a stand of ancient trees. How much value does this scene have? Field economists rate, monetarily, the esthetic value of natural features before parkland is developed. Why is one area valued more than another? It might be the number or age of the trees, the proximity to a waterfall or hoodoos, or the presence of protected or endangered wildlife.

- Choose natural features in your neighbourhood.
- Work with a partner or in a small group to decide what maximum and minimum values you would place on these features.
- Develop a system of linear inequalities to model the features.
- Develop an objective function that represents the value of the features.
- Share your model with another pair or group. Justify the decisions you made when developing your model.