6.1 Graphing Linear Inequalities in Two Variables

YOU WILL NEED

- graphing technology OR graph paper, ruler, and coloured pencils

EXPLORE…

- For which inequalities is (3, 1) a possible solution? How do you know?

a) $13 - 3x > 4y$
b) $2y - 5 \leq x$
c) $y + x < 10$
d) $y \geq 9$

solution set

The set of all possible solutions.

GOAL

Solve problems by modelling linear inequalities in two variables.

INVESTIGATE the Math

Amir owns a health-food store. He is making a mixture of nuts and raisins to sell in bulk. His supplier charges $25/kg for nuts and $8/kg for raisins.

? What quantities of nuts and raisins can Amir mix together if he wants to spend less than $200 to make the mixture?

A. Suppose that Amir wants to spend exactly $200 to make the mixture. Work with a partner to create an equation that represents this situation.

B. To what set of numbers does the domain and range of the two variables in your equation belong? Use this information to help you graph the equation on a coordinate plane.

C. Explain why the graph is a line segment, not a ray or a line.

D. What region of the coordinate plane includes points representing quantities of nuts and raisins that Amir could use if he wants to spend less than $200? How do you know?

E. There are many possible solutions to Amir's problem. Plot at least three points that represent reasonable solutions to Amir's problem. Explain why you chose these points.

Reflecting

F. Discuss and then decide whether the **solution set** for Amir's problem is represented by

i) points in the region above the line segment.
ii) points in the region below the line segment.
iii) points on the line segment.

G. Why might the line segment be considered a boundary of the solution set?

H. Why might you use a dashed line segment for this graph instead of a solid line segment?

APPLY the Math

EXAMPLE 1 Solving a linear inequality graphically when it has a continuous solution set in two variables

Graph the solution set for this linear inequality:
$-2x + 5y \geq 10$

Robert's Solution: Using graph paper

Linear equation that represents the boundary:
$-2x + 5y = 10$

I knew that the graph of the linear equation $-2x + 5y = 10$ would form the boundary of the linear inequality $-2x + 5y \geq 10$.

The variables represent numbers from the set of real numbers.
$x \in \mathrm{R}$ and $y \in \mathrm{R}$

The domain and range are not stated and no context is given, so I assumed that the domain and range are the set of real numbers. This means that the solution set is **continuous**.

continuous

A connected set of numbers. In a continuous set, there is always another number between any two given numbers. Continuous variables represent things that can be measured, such as time.

y-intercept:

$$-2x + 5y = 10$$
$$-2(0) + 5y = 10$$
$$\frac{5y}{5} = \frac{10}{5}$$
$$y = 2$$

The y-intercept is at (0, 2).

I knew that I needed to plot and join only two points to graph the linear equation. I decided to plot the two intercepts.

To determine the y-intercept, I substituted 0 for x.

x-intercept:

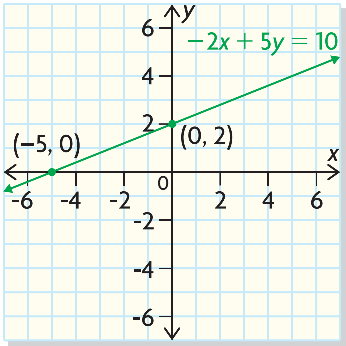
$$-2x + 5y = 10$$
$$-2x + 5(0) = 10$$
$$\frac{-2x}{-2} = \frac{10}{-2}$$
$$x = -5$$

The x-intercept is at $(-5, 0)$.

To determine the x-intercept, I substituted 0 for y.

solution region

The part of the graph of a linear inequality that represents the solution set; the solution region includes points on its boundary if the inequality has the possibility of equality.



Since the linear inequality has the possibility of equality (≥), and the variables represent real numbers, I knew that the **solution region** includes all the points on its boundary. That's why I drew a solid green line through the intercepts.

half plane

The region on one side of the graph of a linear relation on a Cartesian plane.

Test (0, 0) in $-2x + 5y \geq 10$.

LS	RS
$-2(0) + 5(0)$	10
0	

Since 0 is not greater than or equal to 10, (0, 0) is not in the solution region.

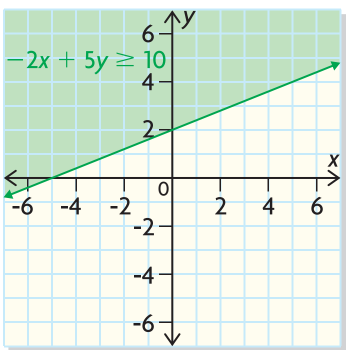
I needed to know which **half plane**, above or below the boundary, represents the solution region for the linear inequality.

To find out, I substituted the coordinates of a point in the half plane below the line. I used (0, 0) because it made the calculations simple.

I already knew that the solution region includes points on the boundary, so I didn't need to check a point on the line.

Communication *Tip*

If the solution set to a linear inequality is continuous and the sign includes equality (≤ or ≥), a solid green line is used for the boundary, and the solution region is shaded green, as shown to the right.



Since my test point below the boundary was not a solution, I shaded the half plane that did not include my test point. This was the region above the boundary.

I used green shading to show that the solution set belongs to the set of real numbers.

Since the domain and range are in the set of real numbers, I knew that the solution set is continuous. Therefore, the solution region includes all points in the shaded area and on the solid boundary.

Janet's Solution: Using graphing technology

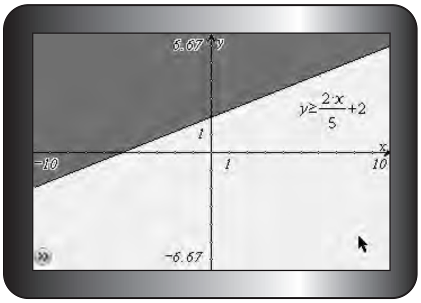
$$-2x + 5y \geq 10$$
$$5y \geq 2x + 10$$
$$\frac{5y}{5} \geq \frac{2x + 10}{5}$$
$$y \geq \frac{2x}{5} + 2$$

To enter the linear inequality into my graphing calculator, I had to isolate *y*.

I didn't have to reverse the inequality sign because I didn't divide or multiply by a negative value.

The variables represent numbers from the set of real numbers.
$x \in \mathrm{R}$ and $y \in \mathrm{R}$

The domain and range are not stated and no context is given, so I assumed the domain and range are in the set of real numbers. This means that the solution set is continuous.



I entered the inequality into the calculator and graphed it.

I knew that the boundary was correctly shown because a solid line means equality is possible.

Test (1, 4) in $-2x + 5y \geq 10$.

LS	RS
$-2(1) + 5(4)$	10
18	

To verify whether the correct half plane was shaded, I used (1, 4) in the half plane above the boundary as a test point.

Since (1, 4) is a solution to the linear inequality, I knew that the half plane above the boundary should be shaded.

Since 18 is greater than 10, (1, 4) is in the solution region.

The solution region includes all the points in the shaded area and along the boundary because the solution set is continuous.

Your Turn

Compare the graphs of the following relations. What do you notice?

$-2x + 5y \geq 10$ $\qquad$ $-2x + 5y = 10$ $\qquad$ $-2x + 5y < 10$

EXAMPLE 2 Graphing linear inequalities with vertical or horizontal boundaries

Graph the solution set for each linear inequality on a Cartesian plane.

a) $\{(x, y) \mid x - 2 > 0, x \in \mathrm{R}, y \in \mathrm{R}\}$

b) $\{(x, y) \mid -3y + 6 \geq -6 + y, x \in \mathrm{I}, y \in \mathrm{I}\}$

Wynn's Solution

a) $x - 2 > 0$

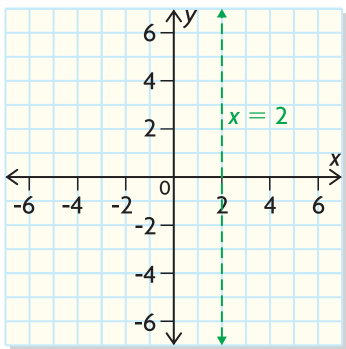
$x > 2$

I isolated *x* so I could graph the inequality.

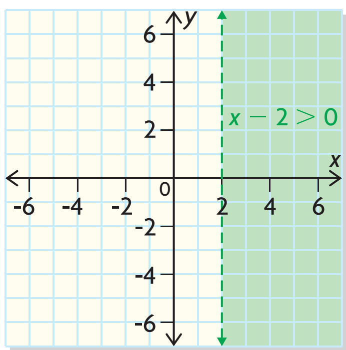
The variables represent numbers from the set of real numbers.

$x \in \mathrm{R}$ and $y \in \mathrm{R}$

The domain and range are stated as the set of real numbers. The solution set is continuous, so the solution region and its boundary will be green in my graph.



I drew the boundary of the linear inequality as a dashed green line because I knew that the linear inequality (>) does not include the possibility of *x* being equal to 2.



I needed to decide which half plane to shade. For *x* to be greater than 2, I knew that any point to the right of the boundary would work.

The solution region includes all the points in the shaded area because the solution set is continuous. The solution region does not include points on the boundary.

Communication *Tip*

If the solution set to a linear inequality is continuous and the sign does not include equality (< or >), a dashed green line is used for the boundary and the solution region is shaded green, as shown to the right.

b) $-3y + 6 \geq -6 + y$

$$-4y \geq -12$$

$$\frac{-4y}{-4} \leq \frac{-12}{-4}$$

$$y \leq 3$$

Since the linear inequality has only one variable, y, I isolated the y.

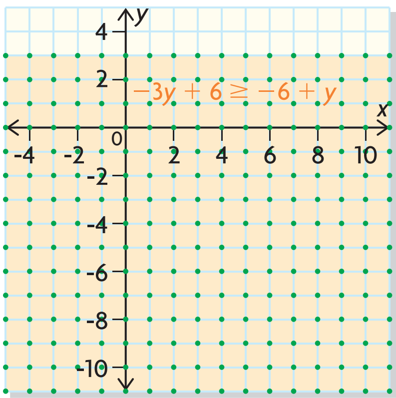
As I rearranged the linear inequality, I divided both sides by -4. That's why I reversed the sign from $\geq$ to $\leq$.

The variables represent integers.
$x \in$ I and $y \in$ I

The domain and range are stated as being in the set of integers. I knew this means that the solution set is **discrete**.

discrete

Consisting of separate or distinct parts. Discrete variables represent things that can be counted, such as people in a room.



I knew that points with integer coordinates below the line
$y = 3$
were solutions, so I shaded the half plane below it orange.

I knew the linear inequality includes 3, so points on the boundary with integer coordinates are also solutions to the linear inequality.

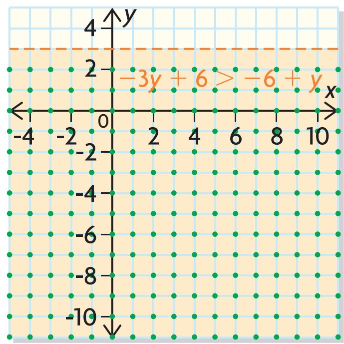
I stippled the boundary and the orange half plane with green points to show that the solution set is discrete.

$\{(x, y) \mid -3y + 6 \geq -6 + y, x \in \text{I}, y \in \text{I}\}$

The solution region includes only the points with integer coordinates in the shaded region and along the boundary.

Communication *Tip*

If the solution set to a linear inequality is discrete, the solution region is shaded orange and stippled with green points. If the sign includes equality ($\geq$ or $\leq$), the boundary is also stippled. An example of this is shown to the left. If equality is not possible ($<$ or $>$), the boundary is a dashed orange line. An example of this is shown below.



Your Turn

How can you tell if the boundary of a linear inequality is vertical or horizontal without graphing the linear inequality? Explain.

EXAMPLE 3 Solving a real-world problem by graphing a linear inequality with discrete whole-number solutions

A sports store has a net revenue of \$100 on every pair of downhill skis sold and \$120 on every snowboard sold. The manager's goal is to have a net revenue of more than \$600 a day from the sales of these two items. What combinations of ski and snowboard sales will meet or exceed this daily sales goal? Choose two combinations that make sense, and explain your choices.

Jerry's Solution

The relationship between the number of pairs of skis, x, the number of snowboards, y, and the daily sales can be represented by the following linear inequality:

$100x + 120y > 600$

> I defined the variables in this situation and wrote a linear inequality to represent the problem.

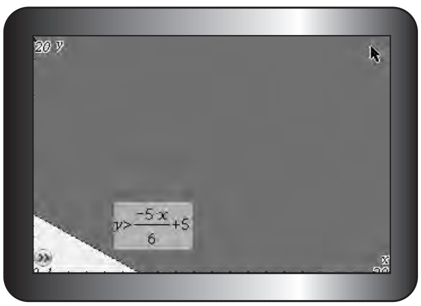
The variables represent whole numbers.

$x \in W$ and $y \in W$

> I knew that only whole numbers are possible for x and y, since stores don't sell parts of skis or snowboards.
>
> Because the domain and range are restricted to the set of whole numbers, I knew that the solution set is discrete.
>
> I also knew that my graph would occur only in the first quadrant.

$$100x + 120y > 600$$
$$120y > 600 - 100x$$
$$\frac{120y}{120} > \frac{600 - 100x}{120}$$
$$y > \frac{600}{120} - \frac{100x}{120}$$
$$y > 5 - \frac{5x}{6}$$
$$y > -\frac{5x}{6} + 5$$

> I isolated y so I could enter the inequality into my graphing calculator.



> I adjusted the calculator window to show only the first quadrant, since the domain and range are both the set of whole numbers.
>
> The boundary is a dashed line, which means that the solution set does not include values on the line.

$\{(x, y) \mid 100x + 120y > 600, x \in W, y \in W\}$

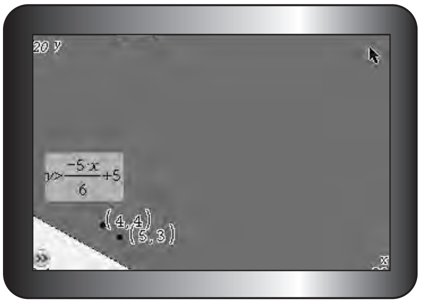
Test (0, 0) in $100x + 120y > 600$.

LS	RS
$100(0) + 120(0)$	600
0	

Since 0 is not greater than 600, (0, 0) is not in the solution region.

I used the test point (0, 0) to verify that the correct half plane was shaded.

Since (0, 0) is not a solution to the linear inequality, I knew that the half plane that did not include this point should be shaded. This was done correctly.



When I interpreted the graph, I considered the context of the problem. I knew that

- only discrete points with whole-number coordinates in the solution region made sense.
- points along the dashed boundary are not part of the solution region.
- points with whole-number coordinates along the *x*-axis and *y*-axis boundaries are part of the solution region.

Test (4, 4) in $100x + 120y > 600$.

LS	RS
$100(4) + 120(4)$	600
$400 + 480$	
880	

Since $880 > 600$, (4, 4) is a solution.

I picked two points in the solution region, (4, 4) and (5, 3), as possible solutions to the problem. I verified that each point is a solution to the linear inequality.

Test (5, 3) in $100x + 120y > 600$.

LS	RS
$100(5) + 120(3)$	600
$500 + 360$	
860	

Since $860 > 600$, (5, 3) is a solution.

Sales of four pairs of skis and four snowboards or sales of five pairs of skis and three snowboards will exceed the manager's net revenue goal of more than $600 a day.

Some points in the solution region are more reasonable than others. For example, the point (1000, 1000) is a valid solution, but it might be an unrealistic sales goal.

Your Turn

a) Would raising the daily sales goal to at least $1000 change the graph that models this situation? Explain.

b) State two combinations of ski and snowboard sales that would meet or exceed this new daily sales goal.

In Summary

Key Idea

- When a linear inequality in two variables is represented graphically, its boundary divides the Cartesian plane into two half planes. One of these half planes represents the solution set of the linear inequality, which may or may not include points on the boundary itself.

Need to Know

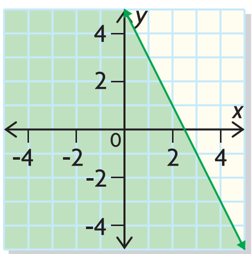
- To graph a linear inequality in two variables, follow these steps:

  Step 1. Graph the boundary of the solution region.
  - If the linear inequality includes the possibility of equality ($\leq$ or $\geq$), and the solution set is continuous, draw a solid green line to show that all points on the boundary are included.
  - If the linear inequality includes the possibility of equality ($\leq$ or $\geq$), and the solution set is discrete, stipple the boundary with green points.
  - If the linear inequality excludes the possibility of equality ($<$ or $>$), draw a dashed line to show that the points on the boundary are not included.
    - Use a dashed green line for continuous solution sets.
    - Use a dashed orange line for discrete solution sets.

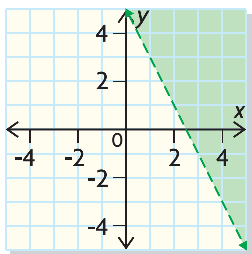
  Step 2. Choose a test point that is on one side of the boundary.
  - Substitute the coordinates of the test point into the linear inequality.
  - If possible, use the origin, (0, 0), to simplify your calculations.
  - If the test point is a solution to the linear inequality, shade the half plane that contains this point. Otherwise, shade the other half plane.
    - Use green shading for continuous solution sets.
    - Use orange shading with green stippling for discrete solution sets.

For example,

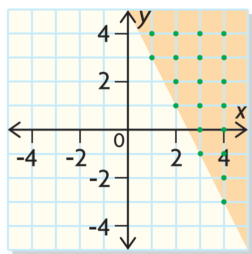
$\{(x, y) \mid y \leq -2x + 5, x \in R, y \in R\}$



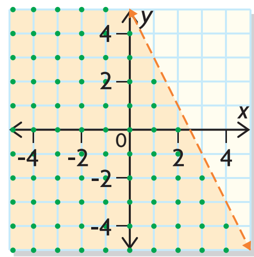
$\{(x, y) \mid y > -2x + 5, x \in R, y \in R\}$



$\{(x, y) \mid y \geq -2x + 5, x \in I, y \in I\}$



$\{(x, y) \mid y < -2x + 5, x \in I, y \in I\}$



- When interpreting the solution region for a linear inequality, consider the restrictions on the domain and range of the variables.
  - If the solution set is continuous, all the points in the solution region are in the solution set.
  - If the solution set is discrete, only specific points in the solution region are in the solution set. This is represented graphically by stippling.
  - Some solution sets may be restricted to specific quadrants. For example, most linear inequalities representing real-world problem situations have graphs that are restricted to the first quadrant.

CHECK Your Understanding

1. Graph the solution set for each linear inequality.
a) $y < x + 4$ **b)** $-y < -6x + 3$

2. Consider the graph of this inequality.

$$2x + 3y > 5$$

Make each of the following decisions, and provide your reasoning.
a) whether the boundary should be dashed, stippled, or solid
b) whether the half plane above or below the boundary should be shaded
c) whether each point is in its solution region:
i) (1, 1) **ii)** (1, 0) **iii)** (1, 2)

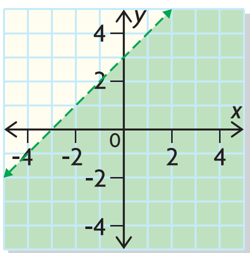
3. Betsy and Flynn work at an ice cream stand. If Betsy worked three times as many hours as she usually does and Flynn worked twice the number of hours that he usually does, together they would work less than 25 h. The situation can be modelled by the following linear inequality:

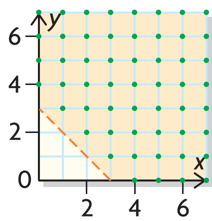
$$3b + 2f < 25$$

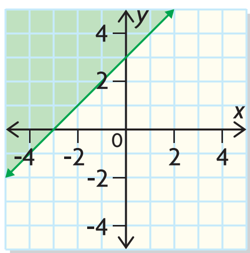
a) What do the variables b and f represent?
b) What restrictions does the context place on the variables? Explain.
c) Suppose you were to graph the inequality.
i) Describe the boundary.
ii) Would you shade the half plane above or below the boundary?
iii) Would your graph involve all four quadrants? Explain.
d) What does a solution to this inequality represent?

PRACTISING

4. Match each graph with its linear inequality, and justify your match.

a)


b)


c)


i) $\{(x, y) \mid x - 3 > -y, x \in W, y \in W\}$
ii) $\{(x, y) \mid x - y > -3, x \in R, y \in R\}$
iii) $\{(x, y) \mid y - 3 \geq x, x \in R, y \in R\}$

5. Graph the solution set for each linear inequality.
a) $y > -2x + 8$ **d)** $-4x - 8 > 4$
b) $-3y \leq 9x + 12$ **e)** $10x - 12 < -y$
c) $y < 6$ **f)** $4x + 3y \geq -12$

6. Graph the solution set for each linear inequality.
   a) $\{(x, y) \mid 2x - y \geq 5y + 2x + 12, x \in W, y \in W\}$
   b) $\{(x, y) \mid x + 6y - 14 < 0, x \in I, y \in I\}$
   c) $\{(x, y) \mid 5x - y \leq 4, x \in W, y \in W\}$
   d) $\{(x, y) \mid 2x + 2 \leq 5 + x, x \in I, y \in I\}$
   e) $\{(x, y) \mid -2y > 20, x \in R, y \in R\}$
   f) $\{(x, y) \mid 4x - 5y < 10, x \in R, y \in R\}$

7. Grace's favourite activities are going to the movies and skating with friends. She budgets herself no more than \$75 a month for entertainment and transportation. Movie admission is \$9 per movie, and skating costs \$5 each time. A student bus pass for the month costs \$25.
   a) Define the variables and write a linear inequality to represent the situation.
   b) What are the restrictions on the variables? How do you know?
   c) Graph the linear inequality. Use your graph to determine:
      i) a combination of activities that Grace can afford and still have some money left over
      ii) a combination of activities that she can afford with no money left over
      iii) a combination of activities that will exceed her budget

8. Eamon coaches a hockey team of 18 players. He plans to buy new practice jerseys and hockey sticks for the team. The supplier sells practice jerseys for \$50 each and hockey sticks for \$85 each. Eamon can spend no more than \$3000 in total. He wants to know how many jerseys and sticks he should buy.
   a) Write a linear inequality to represent the situation.
   b) Use your inequality to model the situation graphically.
   c) Determine a reasonable solution to meet the needs of the team, and provide your reasoning.

9. For every teddy bear that is sold at a fundraising banquet, \$10 goes to charity. For every ticket that is sold, \$32 goes to charity. The organizers' goal is to raise at least \$5000. The organizers need to know how many teddy bears and tickets must be sold to meet their goal.
   a) Define the variables and write a linear inequality to represent the situation.
   b) What are the restrictions on the variables? How do you know?
   c) Graph the linear inequality to help you determine whether each of the following points is in the solution set. The first coordinate is the number of teddy bears and the second is the number of tickets.
      i) (400, 20) ii) (205, 98) iii) (156, 105)

10. On Earth Day, a nursery sold more than \$1500 worth of maple and birch trees. The maple trees were sold for \$75, and the birch trees were sold for \$50.

a) Define the variables and write a linear inequality to represent the possible combinations of trees sold. Are there any restrictions on the variables? Explain.

b) Graph the linear inequality.

c) Use your graph to determine:

i) if the nursery could have sold 13 of each type of tree

ii) if 14 of one type and 9 of the other type could have been sold

11. In the fall, Javier plants tulip and crocus bulbs. Each tulip takes up an area of at least 12 square inches, and each crocus takes up an area of at least 9 square inches. Javier has a total area of 36 in. by 50 in., and he wants to plant at least 30 of each type of flower. He wants to know exactly how many of each type of flower he should plant.

a) If you were to graph Javier's situation, would the boundary be a dashed line, a stippled line, or a solid line? How do you know?

b) Graph the linear inequality, and determine a reasonable solution to Javier's problem.

12. A banquet room is set up to seat, at most, 660 people. Each rectangular table seats 12 people, and each circular table seats 8 people.

a) Define the variables and write a linear inequality to represent the number of each type of table needed. Then graph your inequality.

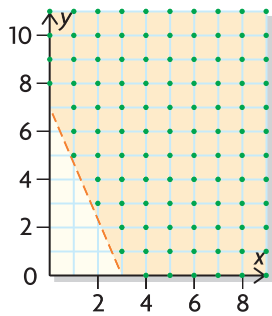
b) The organizers of the banquet would like to have as close to the same number of rectangular tables and circular tables as possible. What combination of tables could they use? Explain your choice.

Closing

13. Joelle used deductive reasoning to conclude that the graph on the right represents a linear inequality.

a) What evidence has she used to arrive at this conclusion?

b) State some other things you know about this inequality. Provide your reasoning for each.



Extending

14. Debbie and Gavin are moving. They have household goods that occupy a volume of, at most, 162 cubic feet. Packing boxes are available in two sizes: 4 cubic feet and 6 cubic feet. They are sold in sets of four.

a) Use a graph to determine what combinations of boxes Debbie and Gavin could buy.

b) If Debbie and Gavin wanted to use the least number of boxes, what is the best combination? Explain your thinking.